BLACKIE & SON LIMITED
66 Chandos Place, LONDON
17 Stanhope Street, GLASGOW

BLACKIE & SON (INDIA) LIMITED
103/5 Fort Street, BOMBAY

BLACKIE & SON (CANADA) LIMITED
TORONTO

THE CLASSICAL THEORY OF ELECTRICITY AND MAGNETISM

BY

MAX ABRAHAM

Formerly Professor of Rational Mechanics at Milan

REVISED BY

RICHARD BECKER

Professor of Physics at the ~~Technische~~ Hochschule, Berlin

Authorized Translation by

JOHN DOUGALL, M.A., D.Sc., F.R.S.E.

BLACKIE & SON LIMITED
LONDON AND GLASGOW

BOOK
PRODUCTION
WAR ECONOMY
STANDARD

THE PAPER AND BINDING OF THIS BOOK CONFORM TO THE AUTHORIZED ECONOMY STANDARDS

First published 1932
Reprinted 1937, 1942, 1943, 1944
1946, 1947

Printed in Great Britain by Blackie & Son, Ltd., Glasgow

PREFACE

The present work is a translation of the eighth German edition. The notation for vectors has been modified, so as to bring it into line with that customary in English books. A new feature is the collection of problems and solutions. Special thanks are due to the translator for the capable and careful way in which he has carried out his task.

R. BECKER.

BERLIN, *May*, 1932.

The English publishers wish to express their thanks to the authorities of the University of London for permission to use their examination papers in the collection of examples.

PREFACE
TO THE EIGHTH GERMAN EDITION

The work entitled *An Introduction to Maxwell's Theory*, by A. Föppl, appeared in 1894. Ten years later, the second edition, recast and thoroughly revised, was published as the first volume of Max Abraham's *Theory of Electricity*. For a whole generation of physicists after that date, " Abraham-Föppl " was more widely used than any other textbook introductory to electrical theory. The fact that as many as seven editions appeared in Abraham's lifetime is convincing evidence of the estimation in which the work was held by teachers and students.

In the new edition, I have felt bound to preserve the essential features of a book so obviously suited to its purpose, and many passages have been taken over unchanged. At the same time, some fairly extensive alterations have been made in particular sections, always in the direction of laying greater emphasis on the concrete physical content of the theory, and less on its purely formal aspects. To assist the student towards a vivid comprehension of the text, the number of diagrams has been increased more than fivefold.

New sections have been added dealing with electrostriction, and with the thermodynamics of the field. The theory of the skin effect has been amplified, and the theory of waves in wires has been extended to the case when resistance is taken into account. In the exposition of the theory of alternating currents, advantage has been taken of the vector diagram used by electrical engineers. The treatment of electric currents as a cyclic system has been omitted altogether. The substance of the last two sections of the previous edition—on ferromagnetism and induction phenomena in moving bodies—has been incorporated in other sections.

In the choice of units I have followed Abraham's last edition in every detail. The system used throughout is the Gaussian system, in which the energy density in a vacuum is equal to

$$\frac{1}{8\pi}\left(\mathbf{E}^2 + \mathbf{H}^2\right) \text{ ergs/cm.}^3,$$

and the dielectric constant and permeability of a vacuum are each taken as unity. It does not seem possible at present to set up a system of units which will satisfy the electrical engineer and the physicist alike. With regard to Maxwell's theory, the difference between the physicist and the electrician is not a matter of notation merely, but of principle. The technical view adheres much more strictly than current physics does to the original form of the Faraday-Maxwell theory. The engineer looks upon the vectors **E** and **D** even in a vacuum as magnitudes of quite different kinds, related to one another more or less like tension and extension in the theory of elasticity. From this point of view it must of course seem a very questionable procedure, in an exposition of fundamental principles, to put the factor of proportionality K, in the equation $\mathbf{D} = K\mathbf{E}$, equal to 1 for empty space, thus artificially attributing to **D** and **E** the same dimensions. On the other hand, the distinction in principle between **D** and **E**, which is closely connected with the mechanical theory of the æther, has been absolutely abandoned in modern physics, the electromagnetic conditions at any point in empty space being now regarded as completely defined when we are given *one* electric vector **E** and *one* magnetic vector **B** (or **H**). The numerical identity of **E** and **D** (for empty space) in the Gaussian system of units is not, for the physicist, the result of an arbitrary definition, but the expression of the fact that **E** and **D** are actually the same thing. The introduction by the engineer of a dielectric constant and permeability not equal to 1 in a vacuum seems to the physicist to be merely an artifice, by means of which formulæ are reduced to a shape which is convenient for practical calculations.

For purposes of reference, a list of important formulæ is given in an appendix.

R. BECKER.

BERLIN, *February*, 1930.

CONTENTS

Part I

VECTORS AND VECTOR FIELDS

CHAPTER I

Vectors

CHAPTER II

Vector Fields

Part II

THE ELECTRIC FIELD

CHAPTER III

The Electrostatic Field in Free Space

CHAPTER IV

Dielectrics

CHAPTER V

Energy and Mechanical Forces in the Electrostatic Field

CHAPTER VI

The Steady Electric Current

Part III

THE ELECTROMAGNETIC FIELD

CHAPTER VII

Magnetic Vectors

CHAPTER VIII

Electrodynamics of Media at Rest

CHAPTER IX

The Electrodynamics of Quasi-steady Currents

CHAPTER X

Electromagnetic Waves

Part IV

ENERGY AND FORCES IN MAXWELL'S THEORY

CHAPTER XI

Thermodynamics of Field Energy

CHAPTER XII

The Forces in Fields which Vary with the Time

INTRODUCTION

The theory of electric and magnetic phenomena, as it existed before Maxwell, was based on the conception of action at a distance between bodies which are electrified, magnetized, or traversed by electric currents. The only physicist who took a different view was Faraday. But he was not enough of a mathematician to express his ideas in the complete and self-consistent form which would have raised them to the rank of a theory. His method of regarding and describing electrical phenomena was, it is true, a mathematical one, but he did not express himself in terms of the ordinary symbolism of mathematics. This was first done by Maxwell, who translated Faraday's ideas into strict mathematical form, and thus built up a theory which differed essentially from the theory of action at a distance even in its foundations, and still more in its higher developments.

The discoveries of Heinrich Hertz supplied the proof that electromagnetic processes do actually take place in dielectrics, and in particular in free space, and the fundamental ideas of Maxwell's theory have now been accepted by all physicists.

What are the essential characteristics which distinguish Maxwell's theory of field action from the theories of action at a distance?

The essential ideas underlying Maxwell's theory which we shall have to consider are these:

1. The idea that all electric and magnetic action of one body on another separated from it is transmitted through the intervening space, whether that be empty or occupied by matter.

2. That the seat of electric or magnetic energy is to be found not only in the body which is electrified or magnetized, or which is traversed by a current, but also, and to a far greater extent, in the surrounding field.

3. That the electric current in an unclosed conducting circuit is closed, or made complete, by a supplementary "displacement current" in the dielectric, and that this displacement current is connected with the magnetic field strength in the same way as the conduction current.

4. That the flux of magnetic induction has no sources, or, in other words, that "true" magnetism is never found.

5. That light waves are electromagnetic waves.

Maxwell himself stated his equations in terms of quaternions, but only rather incidentally; in essence, his exposition is based on Cartesian methods. With the latter, however, it is difficult to grasp the connexion of the formulæ as a whole. It is much easier to do so when the vector calculus is employed. The trouble it costs to make oneself familiar with vector methods is amply repaid by the advantages gained. The use of vectors is in fact indispensable, if what we aim at is to secure as faithful a reproduction as possible of Faraday's idea of the flux of force. The theory of vectors and vector fields is therefore placed at the head of the present work. The notation is that now used by nearly all writers who are doing original work in electrodynamics. In the following chapters, the method of vectors, which is useful in rigid dynamics and in hydrodynamics as well as in electricity, will be employed throughout.

Part I
VECTORS AND VECTOR FIELDS

CHAPTER I
Vectors

1. Definition of a Vector.

The equations of physics are ultimately relations between quantities which are immediately measurable. What a measurement tells is the number of times a given unit is contained in the quantity measured. The unit may be chosen arbitrarily (e.g. a metre, a second, a degree Centigrade), or it may be reduced to other units, previously defined, with which it is connected by an equation. The formula for the unit, obtained by solving this equation, represents the "dimensions" of the unit with respect to the other units. The so-called "absolute" system of measurement employs the three fundamental units of length, mass, and time; but no matter what units are chosen to serve as the foundation of the absolute system, the two sides of any equation in physics must "balance", i.e. they must agree with each other not only numerically but also in dimensions. In fact, if there were any disparity in the dimensions, a change in the fundamental units would destroy the numerical equality of the two sides of the equation. The fact that the dimensions must balance is taken advantage of in physical calculations as a first check upon the accuracy of an equation.

Physical quantities of the simplest type are completely defined by the assignment of a single number, along with a known unit. Such quantities are called *scalars*; mass and temperature are examples.

But there are other physical quantities, which do not belong to the class of scalars. Thus, in order to specify the final position of a point which is displaced from a given initial position, three numbers are required, say, for example, the Cartesian co-ordinates of the final point with respect to axes through the initial point. In this case we might, without introducing any new kind of quantity, work throughout with scalars, viz. the component displacements. But if we did so we should in the first place be neglecting the fact that, physically speaking,

a displacement is a single idea; and, secondly, we should be importing a foreign element into the question, viz. the co-ordinate system, which has nothing to do with the displacement itself. We shall therefore introduce displacements as quantities of a new type, and establish a system of rules for their use. Only when we come to evaluate formulæ numerically will it be necessary to bring in a definite co-ordinate system.

Rectilinear displacements of a point, and all physical quantities which can be represented by such displacements (in the same way as the values of a scalar can be represented by the points of a straight line), and which also obey the same law of addition as the corresponding displacements, are called *vectors*.

A rigorous test for determining whether a quantity is a vector or not will be given in § 3 (p. 6).

2. Addition and Subtraction of Vectors.

In the definition of a vector just given, vector addition has been reduced to composition of rectilinear displacements. Take now two vectors **A*** and **B** of the same dimensions and type; in order to add them, consider a movable point situated to begin with at 1 (fig. 1). Let this point be given, first, the displacement (1, 2), representing the vector **A** in magnitude, direction, and sense; then the displacement (2, 3), agreeing in length, direction, and sense † with the vector **B**; the result is equivalent to a displacement of the movable point from 1 to 3. This rectilinear displacement which takes the point directly from 1 to 3 is called the *resultant* or *geometric sum* of the two displacements (1, 2) and (2, 3). It represents a vector **C** which, in accordance with the definition of § 1, we call the resultant or sum of the vectors **A** and **B**:

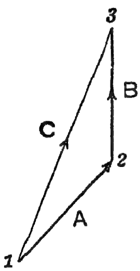

Fig. 1

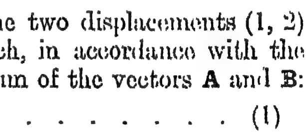

Fig. 2

$$\mathbf{C} = \mathbf{A} + \mathbf{B}. \qquad \qquad (1)$$

If the displacement **B** is made first, and then the displacement **A** (fig. 2), the movable point describes the path (143), which with (123) makes up a parallelogram; accordingly the resultant of the displacements **B** and **A**, like that of **A** and **B**, is represented by the diagonal (1, 3) of that parallelogram. Hence vector addition obeys the *commutative law*: *the geometric sum of two vectors is independent of the order of addition*:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \qquad \qquad (2)$$

* Heavy type will be used throughout to indicate vectors.

† In future, the word *direction* will be used so as to include *sense*. [Tr.]

This parallelogram law of addition (fig. 2) is characteristic of the quantities called vectors. Quantities exist with which we can associate the properties of magnitude and direction, but which follow another law of composition. For example, we know from kinematics that infinitely small rotations of a rigid system about a fixed point can be represented by vectors, since their composition obeys the parallelogram law. On the other hand finite rotations are compounded in a more complicated way, and therefore are not vectors. It is proved in statics that forces acting on a particle follow the parallelogram law of addition; such forces are therefore vectors.

If we consider the displacements derived by addition from three vectors **A**, **B**, and **C**, we see that the following law holds, called the *associative law of vector addition*:

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}). \quad . \quad . \quad . \quad . \quad (3)$$

In fig. 3 the sum of three vectors is found by completing the quadrilateral, which has the individual vectors and their sum for its sides; and similarly the sum of n vectors is formed by means of the so-called vector polygon; this has $n + 1$ sides, namely the n vectors which are to be added, and their resultant.

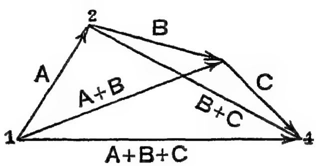

Fig. 3

We may now ask the question: What meaning is to be attached to the *geometric difference* of two vectors **A** and **B**? The answer is that we define the difference in such a way that the relation

$$\mathbf{B} - \mathbf{B} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

holds for vectors, just as the similar relation holds for scalars. The vector $-\mathbf{B}$ therefore corresponds to a displacement which annuls the displacement **B**, i.e. brings the movable point back to its original position. Thus $-\mathbf{B}$ is a vector of the same magnitude as **B**, but in the opposite direction. By the geometric difference of the vectors **A** and **B** we mean the geometric sum of **A** and $-\mathbf{B}$, so that we define *vector subtraction* as follows:

A vector **B** *is subtracted from a vector* **A** *by adding to* **A** *a vector of the same magnitude as* **B**, *but in the opposite direction.*

In the parallelogram of fig. 2, the diagonal (13) represents the geometric sum $\mathbf{A} + \mathbf{B}$, the diagonal (42) the geometric difference $\mathbf{A} - \mathbf{B}$.

The rules for the addition and subtraction of vectors which have now been laid down agree formally with the laws of ordinary algebra.

3. Unit Vectors and Fundamental Vectors. Components.

By the *product* **A** *of a scalar* α *and a vector* **a**,

$$\mathbf{A} = \alpha\mathbf{a} = \mathbf{a}\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

we understand a vector whose magnitude is equal to the product of the magnitudes of the scalar α and the vector **a**,

$$|\mathbf{A}| = |\alpha| \,.\, |\mathbf{a}|, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

and which has the same direction as **a**, or the opposite direction, according as the scalar α is positive or negative.

The multiplication of vectors by scalars obeys the rules of the algebra of scalar quantities. The commutative law has already been explicitly stated in (5); and the distributive law also holds, i.e.

$$(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}, \quad \alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}. \quad . \quad . \quad (5b)$$

All vectors **A** which have the same direction can be connected with a vector **s**, also in that direction, and of magnitude 1:

$$\mathbf{A} = |\mathbf{A}|\,\mathbf{s}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

A vector **s**, of magnitude 1, is called a *unit vector*. We shall adopt the convention of associating the dimensions (§ 1) of a vector with its magnitude; the unit vector **s** in (6) must therefore be given the dimensions of a pure number. Unit vectors afford a convenient means of specifying the direction of a vector, or of a number of parallel vectors.* Let there now be given a fixed unit vector **s**, and an arbitrary vector **a**, which makes with **s** the angle ϕ. The quantity

$$a_s = |\mathbf{a}| \cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

is called the *component of* **a** *relative to the unit vector* **s**, or the *component of* **a** *in the direction* **s**; it is equal to the length of the projection of **a** on the line of the unit vector **s**, taken with the positive or negative sign according as the projection agrees in sense with **s** or not.

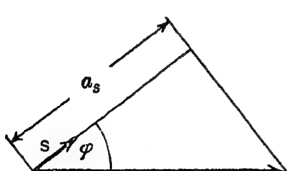

Fig. 4

The component of a vector is a scalar quantity; if we wish to express the projection of **a** on the line of the unit vector **s** in a form which indicates its direction also, we have to multiply the component of **a** in the direction **s** by the unit vector **s** itself: hence the projection as a vector is represented (fig. 4) by

$$|\mathbf{a}| \cos\phi \,.\, \mathbf{s}.$$

* The word *direction* is sometimes used as equivalent to *unit vector*. [Tr.]

Consider next the sum of three vectors **A**, **B**, **C**:

$$\mathbf{A} + \mathbf{B} + \mathbf{C}.$$

As may be seen from fig. 5, the component of this sum in the direction **s** is

$$A_s + B_s + C_s,$$

i.e. the algebraic sum of the components of the vectors **A**, **B**, **C** in the direction **s**. This result can be extended to any number of vectors, so that we have the theorem: *the component of the geometric sum of any number of vectors in a given direction is equal to the algebraic sum of the corresponding components of the vectors taken separately.*

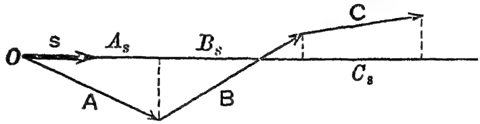

Fig. 5

Vectors of arbitrary direction and magnitude can be specified by their components in fixed directions. For this purpose we require three directions (or unit vectors) not lying in one plane. We choose three mutually perpendicular unit vectors **i**, **j**, **k** as "*fundamental vectors*"; their directions may be taken as those of the axes of a Cartesian co-ordinate system.

As we know, systems of axes **x**, **y**, **z** are of two kinds, which are distinguished as right-handed and left-handed; all right-handed systems can be brought into coincidence with one another, and all left-handed systems with one another, but not a right-handed system with a left-handed system. By reflection in one of the co-ordinate planes a right-handed system becomes a left-handed system, and inversely. Also, by reflection in the origin of co-ordinates (reversal of the directions of all three axes), a right-handed system becomes a left-handed system, and inversely. Following Maxwell, we shall in this book always employ the right-handed system.

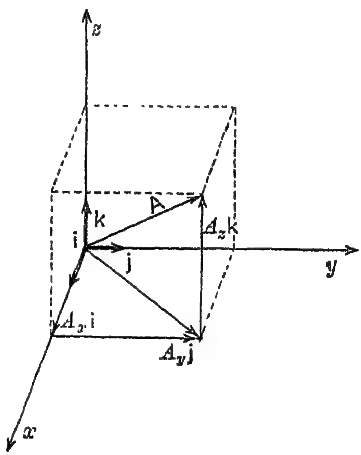

Fig. 6

Such a system is shown in fig. 6. The x-, y-, z-axes, which are also the lines of the fundamental vectors **i**, **j**, **k**, are related to each other in such a way that a rotation from the x-direction to the y-direction, combined with a forward motion in the z-direction, is similar

to the motion of a right-handed screw; so that the x-, y-, z-directions of a right-handed system can be indicated respectively by the thumb, forefinger, and middle finger of the right hand.

Let the components of the vector **a** relative to the fundamental vectors **i**, **j**, **k**, i.e. in the directions of the axes x, y, z, be

$$a_x, \; a_y, \; a_z.$$

Then the projections of the vector **a** on the axes are given in magnitude and direction by

$$a_x\mathbf{i}, \; a_y\mathbf{j}, \; a_z\mathbf{k}.$$

Summation of these three vectors brings us back, as we see from fig. 6, to the vector **a** itself; we have therefore

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Suppose now that the vector **a** is given, in magnitude and direction; then its components are uniquely defined by the equations

$$a_x = |\,\mathbf{a}\,| \cos(\mathbf{a}, x), \quad a_y = |\,\mathbf{a}\,| \cos(\mathbf{a}, y), \quad a_z = |\,\mathbf{a}\,| \cos(\mathbf{a}, z). \quad (8a)$$

Conversely, when the three components are assigned, the vector **a** is uniquely specified as the diagonal of the rectangular parallelepiped, whose edges are the vectors $a_x\mathbf{i}$, $a_y\mathbf{j}$, $a_z\mathbf{k}$. Its magnitude is

$$|\,\mathbf{a}\,| = \sqrt{(a_x^2 + a_y^2 + a_z^2)}, \quad . \quad . \quad . \quad . \quad (8b)$$

and its direction is given by the three cosines, which can be found from (8a) and (8b).

From the three components of a vector relative to the fundamental vectors **i**, **j**, **k** we can calculate its component relative to any unit vector **s**, when we know the angles which **s** makes with the fundamental vectors. According to equation (8), and the theorem illustrated in fig. 5, we have to express the vector **a** as the sum of three partial vectors parallel to **i**, **j**, **k**, and then take the algebraic sum of the components of these three partial vectors in the direction **s**. We thus find

$$a_s = a_x \cos(\mathbf{s}, x) + a_y \cos(\mathbf{s}, y) + a_z \cos(\mathbf{s}, z). \quad . \quad . \quad (9)$$

This law for the calculation of the component of a vector relative to an arbitrary axis is distinctive of vectors. The law expresses the fact that to every direction in space there corresponds a certain scalar quantity, namely the component of the vector **a** in that direction; and that this scalar is a homogeneous linear function of the cosines defining the direction. Conversely, equation (9) gives a general test for determining whether a physical quantity is a vector or not: *by means of a vector, with every direction in space there is associated a scalar "component", which is a homogeneous linear function of the components (i.e. the direction cosines) of the unit vector in that direction.*

4. The Inner or Scalar Product.

Let $\mathbf{F}$ be a force acting at a certain point. If the point moves with velocity $\mathbf{v}$, then the work done by the force per unit time is a scalar quantity; its value is given by the product of the magnitudes of the vectors $\mathbf{F}$ and $\mathbf{v}$ and the cosine of the angle between them. We write this product in the form

$$\mathbf{F}\mathbf{v} = |\mathbf{F}|\,.\,|\mathbf{v}|\,.\cos(\mathbf{F}, \mathbf{v}) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and call it the *scalar product, or* (after Grassmann) *the inner product*, of the two vectors; and the names and notation are used similarly with *any* pair of vectors.

The cosine of the angle between the directions of the two vectors becomes $+1$ when the directions are the same, -1 when they are opposite, and 0 when they are at right angles. Applying this to the fundamental vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we find

$$\mathbf{ij} = \mathbf{jk} = \mathbf{ki} = 0, \text{ but } \mathbf{ii} = \mathbf{jj} = \mathbf{kk} = 1. \quad . \quad . \quad (11)$$

As is immediately obvious from the definition (10), the scalar product remains unchanged when we change the order of the factors. Thus *inner multiplication of two vectors obeys the commutative law.* The scalar product of two vectors $\mathbf{F}$ and $\mathbf{v}$ may also be regarded as the algebraic product of the magnitude of one of them (say $\mathbf{v}$) by the component of the other ($\mathbf{F}$) in the direction of the first. This interpretation leads immediately to the *distributive law of scalar multiplication,*

$$\mathbf{v}\sum_{h=1}^{n}\mathbf{F}_h = \sum_{h=1}^{n}\mathbf{v}\mathbf{F}_h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In fact, according to a theorem proved at fig. 5 (p. 5), the component in any direction of the geometric sum of the vectors $\mathbf{F}_h$ is equal to the algebraic sum of the components of the vectors $\mathbf{F}_h$ in that direction. If, as above, we regard the vectors $\mathbf{F}_h$ as forces, $\mathbf{v}$ as a velocity, then the statement in (12) is: the work done by the resultant force is equal to the algebraic sum of the amounts of work done by the individual forces.

Since the commutative and associative laws hold, it follows that inner multiplication of vectors is carried out by the rules of ordinary algebra. Thus e.g. we have

$$(\mathbf{a} + \mathbf{b})(\mathbf{c} + \mathbf{d}) = \mathbf{ac} + \mathbf{bc} + \mathbf{ad} + \mathbf{bd}. \quad . \quad . \quad (13)$$

If we express the two factors $\mathbf{A}$ and $\mathbf{B}$ of a scalar product by the fundamental vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we find

$$\mathbf{AB} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$

If we multiply out the right-hand side according to ordinary algebraic rules and take account of (11), we find

$$\mathbf{AB} = A_xB_x + A_yB_y + A_zB_z, \quad . \quad . \quad . \quad . \quad (14)$$

a formula which, from the definition of the scalar product and equation (8a), is equivalent to the known formula of analytical geometry

$$\cos(\mathbf{AB}) = \cos(\mathbf{A}, x)\cos(\mathbf{B}, x) + \cos(\mathbf{A}, y)\cos(\mathbf{B}, y) + \cos(\mathbf{A}, z)\cos(\mathbf{B}, z).$$

5. The Outer Product or Vector Product.

Two vectors **A** and **B** (in that order) define a parallelogram and a currency (i.e. a definite sense in which the perimeter of the parallelogram is described). The area of the parallelogram is

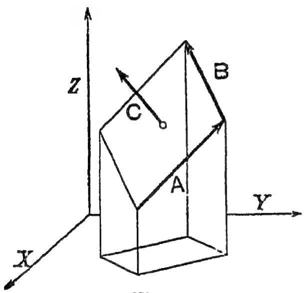

Fig. 7

$$\mathbf{S} = |\mathbf{A}| \,.\, |\mathbf{B}| \,.\, \sin(\mathbf{A}, \mathbf{B}).$$

An entity of this type we shall call a "directed area".* This is defined to be a plane area with a currency. Two directed areas are said to be equal when their planes are parallel, their currencies the same, and their absolute values equal. The directed area defined by the parallelogram which has the two vectors **A** and **B** as consecutive sides, as in fig. 7, is called the *outer product* of the two vectors **A** and **B**; it is denoted by the symbol [**AB**]. With any given directed area we can associate a definite vector **C**, and conversely; viz. we take **C** at right angles to the directed area, and in such a sense that advance in the direction of **C**, and turning in the sense of description of the directed area, constitute together a right-handed screw motion; the length of **C** is equal to the absolute value of the directed area. In the special case of the directed area specified as above by the vectors **A** and **B**, the vector **C** thus defined is called the *vector product* of **A** and **B**, for which we shall write

$$\mathbf{C} = [\mathbf{AB}]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

For calculation with directed areas we now lay down the following definition: *the sum of a number of directed areas is to mean that directed area which is associated with the vector obtained by adding the vectors associated with the given directed areas.*

The appropriateness of this definition can be seen by considering the component C_s of the vector **C** in any direction **s**, which makes with **C** the angle ϕ. In fact, if we project the directed area itself on a plane S perpendicular to **s**,

* Ger. "*Plangrösse*".

the area of the projection has the same numerical value as C_s, viz. $|\mathbf{C}|.|\cos\phi|$. We can also arrange matters so that the projection has the same sign as C_s. We do this by assigning to the plane S the currency which, with **s**, corresponds to a right-handed screw, and then regarding the projection upon S as positive or negative according as the currency of the projection is the same as that of S or not. The component of a directed area, relative to a plane possessing currency, is therefore equal to the component of the vector associated with the directed area, along the (right-handed screw) normal to the plane.

From the above definition it follows that the commutative law does not hold for a vector product. We have in fact

$$[\mathbf{AB}] + [\mathbf{BA}] = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

On the other hand the distributive law is still valid:

$$[(\mathbf{A} + \mathbf{B})\mathbf{D}] = [\mathbf{AD}] + [\mathbf{BD}]. \quad . \quad . \quad . \quad . \quad (17)$$

To prove this equation, we note that a vector product $[\mathbf{AD}]$ is not altered if we replace the vector $\mathbf{A}$ by its projection $\mathbf{A}'$ on a plane perpendicular to $\mathbf{D}$. If this is done with the three vectors $\mathbf{A} + \mathbf{B}$, $\mathbf{A}$, $\mathbf{B}$, and if the figure thus produced in a plane perpendicular to $\mathbf{D}$ has its linear dimensions enlarged in the ratio $|\mathbf{D}| : 1$, and is turned through a right angle, then the vectors $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{A}' + \mathbf{B}'$ become the vectors $[\mathbf{AD}]$, $[\mathbf{BD}]$, and $[(\mathbf{A} + \mathbf{B})\mathbf{D}]$.

In particular, for the unit vectors **i**, **j**, **k**, we have

$$\left.\begin{aligned} [\mathbf{ij}] = \mathbf{k} = -[\mathbf{ji}];\quad [\mathbf{jk}] = \mathbf{i} = -[\mathbf{kj}];\quad [\mathbf{ki}] = \mathbf{j} = -[\mathbf{ik}];\\ \text{and}\quad [\mathbf{ii}] = [\mathbf{jj}] = [\mathbf{kk}] = 0. \end{aligned}\right\} \quad (18)$$

If we use these results to multiply out the vector product

$$[\mathbf{AB}] = [(A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\,(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})],$$

we obtain the formula

$$[\mathbf{AB}] = \mathbf{i}(A_yB_z - A_zB_y) + \mathbf{j}(A_zB_x - A_xB_z) + \mathbf{k}(A_xB_y - A_yB_x), \quad (19)$$

or, in the compact determinant form,

$$[\mathbf{AB}] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad . \quad . \quad (20)$$

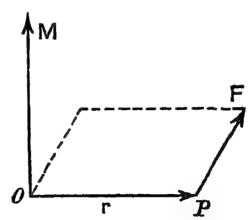

Fig. 8

An example of the outer product of two vectors is the moment of a force **F**. Let O (fig. 8) be the point with respect to which (or about which) moments are taken; and from O let the radius vector **r** be drawn to the point P, at which the force **F** acts. Then the moment vector, i.e. the moment of the force **F** with respect to O, is

$$\mathbf{M} = [\mathbf{rF}]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Another example comes from the kinematics of a rigid body. Let the rigid body have one point O fixed, and let it rotate about the axis ON (fig. 9). Along this axis mark off the segment OU, of a length to indicate the magnitude of the angular velocity, and in the direction having the usual right-handed screw relation to the direction of rotation. We thus associate with the motion of rotation a definite vector **u**. Further, if **r** is the radius vector from O to any point P of the rigid body, and **v** is the velocity of P, then clearly

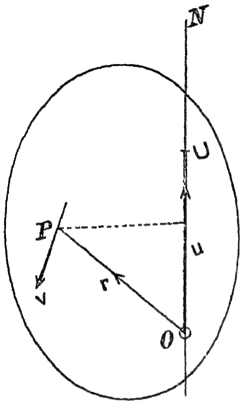

Fig. 9

$$\mathbf{v} = [\mathbf{ur}]. \quad . \quad . \quad . \quad (21a)$$

In fact, the point P moves at right angles to the plane through **r** and **u**; and the arrow, which indicates the direction of **v**, points also in the direction of the vector product. Again, the magnitude of **v** is found by dropping a perpendicular from P to the axis, and multiplying its length by the angular velocity. This gives precisely the absolute value

$$|\,\mathbf{u}\,| \,.\, |\,\mathbf{r}\,| \,.\, \sin(\mathbf{u}, \mathbf{r})$$

of the vector product; the result stated above is therefore proved.

6. Products of Three Vectors.

We shall always use square brackets to denote vector products, and shall therefore employ round brackets when we wish to separate a scalar product of two vectors from other vectors. Three vectors can be multiplied together in three different ways.

(*a*) *Product of a vector and the scalar product of two other vectors*: **A**(**BC**). Here (**BC**) is a scalar, so that **A**(**BC**) is a vector parallel to **A**; and clearly it is a totally different vector from, say, (**AB**)**C**.

(*b*) *Scalar product of a vector and the vector product of two other vectors*: **A**[**BC**]. Here we have the important relation

$$\mathbf{A}[\mathbf{BC}] = \mathbf{B}[\mathbf{CA}] = \mathbf{C}[\mathbf{AB}]. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

In fact, by the elementary rule (volume = base × height) each of these expressions represents the volume of the parallelepiped whose edges are **A**, **B**, **C**. Moreover, all three expressions give this volume with the positive sign, if the vectors **A**, **B**, **C** in that order form a right-handed system.

The expression for the products in terms of the components of the vectors **A**, **B**, **C** is, by (14) and (20),

$$\mathbf{A}[\mathbf{BC}] = \mathbf{B}[\mathbf{CA}] = \mathbf{C}[\mathbf{AB}] = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \qquad (23)$$

(c) *Vector product of a vector and the vector product of two other vectors*: $[\mathbf{A}[\mathbf{BC}]] = \mathbf{F}$.

The vector $\mathbf{F}$ lies in the plane which is defined by the vectors $\mathbf{B}$ and $\mathbf{C}$, and is perpendicular to the projection of $\mathbf{A}$ on this plane. For the vector $[\mathbf{BC}]$ is perpendicular to the plane referred to, and $\mathbf{F}$ is perpendicular to $\mathbf{A}$ and to $[\mathbf{BC}]$.

The x-component of the vector $\mathbf{F}$ is, according to (20),

$$F_x = A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z);$$

this may be written

$$F_x = B_x(A_xC_x + A_yC_y + A_zC_z) - C_x(A_xB_x + A_yB_y + A_zB_z),$$

or, from (14),

$$F_x = B_x(\mathbf{AC}) - C_x(\mathbf{AB});$$

corresponding equations hold for the other components, and we have therefore the single vector equation

$$[\mathbf{A}[\mathbf{BC}]] = \mathbf{B}(\mathbf{AC}) - \mathbf{C}(\mathbf{AB}). \quad . \quad . \quad . \quad . \quad (24)$$

A product of the third kind is by this relation reduced to two products of the first kind. Using the same notation, we also find easily

$$[\mathbf{A}[\mathbf{BC}]] + [\mathbf{B}[\mathbf{CA}]] + [\mathbf{C}[\mathbf{AB}]] = 0, \quad . \quad . \quad . \quad (25)$$

on expanding the terms as in (24).

As one more example of this type, we may calculate the scalar product of two vector products $[\mathbf{AB}] \cdot [\mathbf{CD}]$.

This is a product of the second kind, in which the first vector is replaced by the product of two others; we apply the rule (23), and obtain

$$[\mathbf{AB}] \cdot [\mathbf{CD}] = \mathbf{C}[\mathbf{D}[\mathbf{AB}]].$$

But since, by rule (24), we have

$$[\mathbf{D}[\mathbf{AB}]] = \mathbf{A}(\mathbf{BD}) - \mathbf{B}(\mathbf{AD}),$$

it follows that

$$[\mathbf{AB}] \cdot [\mathbf{CD}] = (\mathbf{AC})(\mathbf{BD}) - (\mathbf{BC})(\mathbf{AD}). \quad . \quad . \quad (26)$$

7. Differentiation of Vectors with Respect to the Time.

The differential coefficient—or derivative—of a vector $\mathbf{a}$ with respect to a scalar variable t (say the time) is defined as a limit by the equation

$$\frac{d\mathbf{a}}{dt} = \lim_{\Delta t = 0} \frac{\mathbf{a}(t + \Delta t) - \mathbf{a}(t)}{\Delta t}. \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Since division by a scalar leaves vectorial properties as they were, the derivative of a vector with respect to a scalar variable is itself a vector. If, for example, $\mathbf{r}$ is the radius vector from a fixed point O to a moving point P, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

gives the velocity vector of the point P.

Since the derivatives of vectors with respect to a scalar variable are deduced by a limiting process from subtraction of vectors and division by scalars, which are operations subject to the rules of ordinary algebra, it follows that the rules of the differential calculus can be extended at once to the differentiation of a sum of vectors:

$$\frac{d(\mathbf{A}+\mathbf{B})}{dt} = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

or of the product of a scalar and a vector:

$$\frac{d\alpha\mathbf{a}}{dt} = \frac{d\alpha}{dt}\mathbf{a} + \alpha\frac{d\mathbf{a}}{dt}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

or of the inner product of two vectors:

$$\frac{d(\mathbf{AB})}{dt} = \left(\frac{d\mathbf{A}}{dt}\mathbf{B}\right) + \left(\mathbf{A}\frac{d\mathbf{B}}{dt}\right). \quad . \quad . \quad . \quad . \quad . \quad (31)$$

A similar rule holds also for the differentiation of an outer product; but in this case care must be taken to write the factors in the correct order:

$$\frac{d}{dt}[\mathbf{AB}] = \left[\frac{d\mathbf{A}}{dt}\mathbf{B}\right] + \left[\mathbf{A}\frac{d\mathbf{B}}{dt}\right]; \quad . \quad . \quad . \quad . \quad (32)$$

since interchange of the factors changes the sign of a vector product.

CHAPTER II

Vector Fields

1. Illustration from Hydrodynamics.

In the first chapter we have developed the idea of a vector, and the rules of vector algebra, with particular reference to the mechanics of a material particle. The velocity of a particle is represented by a single vector. In the present chapter we turn to the problem of analysing the state of motion of a fluid filling space. In this case the velocities of different particles are in general independent of one another, and every point has its own special velocity vector. The moving continuous fluid represents, we say, a *vector field.*

We speak in mathematical physics of the field of a physical quantity, when we consider the quantity from the point of view of its dependence upon position in a region of space; it is assumed that in general (i.e. with the exception of particular surfaces, lines, and points) its values are continuous. A field may be either a scalar field (e.g. a temperature field) or a vector field (e.g. a gravitational field).

The study of fluid motion contributed greatly to the development of the theory of vector fields. Helmholtz's fundamental researches on vortex motion were specially valuable, and were the foundation upon which Maxwell built when he undertook to set Faraday's conception of the field on a mathematical basis. To Maxwell, indeed, hydrodynamical analogies were more than mere mathematical pictures; hydrodynamical ideas with regard to the mechanism of the field guided him in his task of working out a theory of the electromagnetic field as depending upon action transmitted through a medium.

We are therefore following the course of historical development when in this chapter we expound the mathematical theory of vector fields in the light of the hydrodynamical picture. In our previous work with any single vector we have associated a displacement; similarly now we replace the vector whose field we are investigating by the velocity vector of a fluid filling space. In this hydrodynamical representation, it is true, if we do not wish to restrict ourselves to specialized fields we must sometimes suppose the fluid to have properties diverging somewhat from those of actual fluids. There can be no objection to this, since it is only a mathematical analogy which is in question.

2. The Irrotational Field. The Gradient and the Line Integral.

From any scalar field we can derive a vector field as follows. With every point (x, y, z) of space let a scalar ϕ be associated in such a way that $\phi(x, y, z)$ is a continuous and differentiable function of position. In this scalar field we place a small vector ds, and consider the increase of the function ϕ for the change of position ds. If dx, dy, dz are the components of ds, and ds its length, the required increase is

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz. \quad . \quad . \quad . \quad . \quad (1)$$

Since $\dfrac{\partial x}{\partial s} = \cos(s, x)$, &c., it follows that the rate of increase of ϕ per unit length in the direction ds is

$$\frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial x} \cos(s, x) + \frac{\partial \phi}{\partial y} \cos(s, y) + \frac{\partial \phi}{\partial z} \cos(s, z). \quad . \quad (1a)$$

In view of (9), p. 6, this result may be read: The required rate of increase in the direction ds is equal to the component of the vector $\partial\phi/\partial x$, $\partial\phi/\partial y$, $\partial\phi/\partial z$ in the direction ds. We call this vector the *gradient* of the scalar ϕ at the point (x, y, z), and write

$$\mathbf{v} = \operatorname{grad} \phi = \mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z}. \quad . \quad . \quad . \quad (2)$$

The vector $\operatorname{grad}\phi$ is perpendicular to the level surface $\phi = \text{const.}$ For, by (1), the scalar product $(\operatorname{grad} \phi, ds)$ is zero, if the vector ds lies in the surface $\phi = \text{const.}$ The direction of $\operatorname{grad}\phi$ coincides with the direction of steepest ascent of the scalar ϕ. Its magnitude is given by

$$|\operatorname{grad} \phi| = \sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2}.$$

By means of (2) we have therefore derived a vector field $\mathbf{v}(x, y, z)$ from the scalar field $\phi(x, y, z)$.

A conception which is fundamental for the whole of mathematical physics is the *line integral* in a vector field: we join any two points, 1 and 2, of a vector field by an arbitrary curve, which we regard as composed of individual line elements ds (in the direction from 1 to 2). At each point of the curve we form the scalar product

$$(\mathbf{v}\, ds) = v_s ds,$$

where $\mathbf{v}$ is the given vector field, and sum for every ds. On passing to the limit $ds \to 0$, we obtain the line integral

$$\int_1^2 \mathbf{v}\, ds = \int_1^2 v_s ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In general, of course, its value varies with the path of integration.

In the special case before us, where the vector $\mathbf{v}$ *is the gradient of a scalar, the line integral* (3) *is independent of the path joining the points* 1 *and* 2.

In fact,

$$v_s ds = \frac{\partial \phi}{\partial s} ds$$

is the increment of the scalar ϕ for the path element ds; in the line integral (3) all these infinitesimal contributions are added together and give the total increment of ϕ:

$$\int_1^2 \mathbf{v}\, d\mathbf{s} = \phi_2 - \phi_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

The line integral of a gradient has therefore the same value for two paths having the same initial point and the same final point, so that *the line integral of a gradient vanishes for any closed path*:

$$\oint \mathbf{v}\, d\mathbf{s} = \oint \frac{\partial \phi}{\partial s} ds = 0. \quad . \quad . \quad . \quad . \quad . \quad (3b)$$

The fluid motion which represents the field of the vector $\mathbf{v}$ will be called *irrotational* if the line integral of $\mathbf{v}$ over any closed path vanishes, and the field represented will also in this case be called an irrotational field. The above theorem can now be expressed in the form: *the field of the gradient of a scalar* ϕ *is always an irrotational field.*

In a field of force the line integral of the force vector gives the work done. The condition that the line integral along every closed curve should be zero, is here equivalent to the following: it is impossible to obtain an unlimited amount of work by guiding a particle indefinitely often round a closed path. We have proved that this condition is fulfilled when the force vector is the gradient of a scalar.

The converse theorem is also true: *an irrotational vector field can always be regarded as the field of the gradient of a scalar.* In fact, if we assign the arbitrary value ϕ_0 to the scalar at the point O, the value of the scalar at any point P is defined as

$$\phi(P) = \phi_0 + \int_0^P v_s ds, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where, in virtue of (3*b*), the path of integration from O to P is entirely arbitrary. Hence, by giving the final point P of this path the small displacement ds, we obtain

$$d\phi = \mathbf{v}\, d\mathbf{s}, \text{ or } v_s = \frac{\partial \phi}{\partial s}.$$

The irrotational vector field is therefore actually the gradient of the scalar ϕ defined by (4).

If the field we are considering is a field of force, then $(-\phi)$ is called the potential, or preferably the *scalar potential*. The existence of a potential is the necessary and sufficient condition that it should not be possible to obtain an unlimited amount of work from the field of force by the method mentioned above. It was from the irrotational field associated with gravitation that the idea of the potential first arose.

The equation of motion of a material particle in a field of force $\mathbf{F}$, viz.

$$m\dot{\mathbf{v}} = \mathbf{F},$$

gives, after scalar multiplication by $\mathbf{v}$,

$$\frac{d}{dt}(\tfrac{1}{2}mv^2) = (\mathbf{F}\mathbf{v}) = \left(\mathbf{F}\frac{d\mathbf{s}}{dt}\right).$$

Hence $$(\tfrac{1}{2}mv^2)_2 - (\tfrac{1}{2}mv^2)_1 = \int_1^2 \mathbf{F}\,d\mathbf{s}. \quad . \quad . \quad . \quad . \quad (5)$$

If $\mathbf{F} = -\text{grad}\,\phi$, the integral on the right is independent of the path and equal to $\phi_1 - \phi_2$, and we have the result: the increase of kinetic energy is equal to the fall of potential.

The idea of potential has been extended to hydrodynamics, where the scalar $(-\phi)$ associated with every irrotational motion is called the *velocity potential.*

3. The Strength of a Distribution of Sources, Gauss's Theorem, and Divergence.

The ideal fluid on which our hydrodynamical picture is based will in what follows be supposed to be incompressible. This introduces a limitation upon the fluid's freedom of motion, since, in a region which is completely filled, just as much fluid on the whole must enter any closed surface as leaves it. Only special kinds of vector fields could be represented by motion of this type.

In order to remove this limitation, we shall make the additional supposition that at certain points fluid is continually being generated, and at others destroyed. Points of the first sort will be called sources, points of the second sort sinks, or negative sources; but we shall leave ourselves the option of using the word source in a more general sense, as including both positive and negative sources. We are now in a position, by assuming a suitable source system, to represent an arbitrary vector field by a steady motion of an incompressible fluid.

We assume the sources to be continuously distributed in space. The problem then arises of finding a measure for the strength of the source system.

For this purpose, we imagine a definite volume V to be marked off. and we measure the volume of fluid which leaves V per unit time. Since we are assuming the fluid to be incompressible and the motion

steady, this volume of fluid must represent the total strength of all the sources contained within V.

Now the quantity of fluid which in time dt flows in the direction of the (outward or inward) normal across a surface element of area dS is

$$dS \, . \, v_n dt = dS \, |\, \mathbf{v} \,| \, . \cos(\mathbf{v}, n) \, . \, dt,$$

for this is the volume of a cylinder of base dS and height $v_n dt$. The volume of fluid which on the whole flows out of the region V per second is therefore given by the integral taken over the surface S bounding V,

$$\iint v_n dS = \iint \{v_x \cos(n, x) + v_y \cos(n, y) + v_z \cos(n, z)\} \, dS. \quad (6)$$

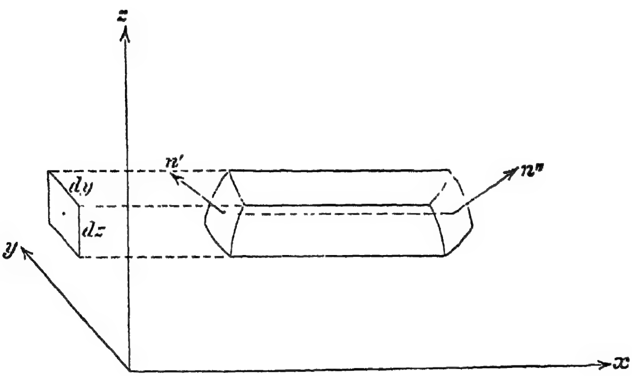

Fig. 1

Here dS is an element of the surface bounding V, and $\mathbf{n}$ is the outward drawn normal. We now convert the integral

$$\iint v_x \cos(n, x) \, dS$$

into a volume integral. For this purpose we suppose V divided up into little prisms of rectangular section parallel to the x-axis. One of these prisms is shown in fig. 1. It has the cross-section $dy\,dz$, and cuts out two elements dS' and dS'' of the surface S. The contribution of these two elements to the surface integral is

$$v_x' \cos(n', x) \, dS' + v_x'' \cos(n'', x) \, dS''.$$

Let x' and x'' be the x-co-ordinates of the ends of the prism, and suppose $x'' > x'$. Then clearly we have

$$\cos(n'', x) \, dS'' = dy\,dz$$

and

$$\cos(n', x) \, dS' = -dy\,dz,$$

so that the contribution of the prism becomes

$$dy\,dz\,.\,(v_x'' - v_x') = dy\,dz\int_{x'}^{x''} \frac{\partial v_x}{\partial x}\,dx.$$

If we now sum over all the prisms into which we supposed V to be divided, we obtain

$$\iint v_x \cos(n, x)\,dS = \iiint \frac{\partial v_x}{\partial x}\,dx\,dy\,dz. \quad . \quad . \quad . \quad (7)$$

Carrying out the corresponding transformation for the other two parts of the right side of (6), we obtain the important result called *Gauss's theorem:*

$$\iint_S v_n\,dS = \iiint_V \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dx\,dy\,dz. \quad . \quad (8)$$

We now define the *divergence* of our field at any point as the outward flow per unit volume from a volume element including the point in question, or

$$\operatorname{div}\mathbf{v} = \lim_{V=0}\left(\frac{1}{V}\iint v_n\,dS\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For its value we find at once from (8)

$$\operatorname{div}\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9a)$$

If $\mathbf{v}(x, y, z)$ is to represent a motion without sources, then $\mathbf{v}$ must everywhere satisfy the differential equation

$$\operatorname{div}\mathbf{v} \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0.$$

4. Green's Theorem.

The possibility of converting a volume integral into a surface integral by Gauss's theorem,

$$\iint_S v_n\,dS = \iiint_V \operatorname{div}\mathbf{v}\,dV,$$

allows us to make some important transformations.

In the first place, let $\mathbf{v}$ be the product of a scalar ψ and a second vector $\mathbf{A}$,

$$\mathbf{v} = \psi\mathbf{A}.$$

Then $$\operatorname{div}\mathbf{v} = \psi\operatorname{div}\mathbf{A} + \frac{\partial \psi}{\partial x}A_x + \frac{\partial \psi}{\partial y}A_y + \frac{\partial \psi}{\partial z}A_z,$$

or $$\operatorname{div}\mathbf{v} = \psi\operatorname{div}\mathbf{A} + (\operatorname{grad}\psi,\ \mathbf{A}),$$

and accordingly, by (8),

$$\iint \psi A_n dS = \iiint \{\psi \operatorname{div} \mathbf{A} + (\operatorname{grad} \psi, \mathbf{A})\} dV. \quad . \quad (10)$$

If further the vector $\mathbf{A}$ can be represented as the gradient of a scalar ϕ ($\mathbf{A} = \operatorname{grad} \phi$), then $A_n = \partial\phi/\partial n$ and

$$\operatorname{div} \mathbf{A} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$

The sum of the second derivatives of a function ϕ is expressed by the notation $\Delta\phi$, where the symbol Δ is called *Laplace's operator*:

$$\Delta\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Hence, when we make the substitution $\mathbf{A} = \operatorname{grad} \phi$, equation (10) takes the form

$$\iint \psi \frac{\partial\phi}{\partial n} dS = \iiint \{\psi \Delta\phi + (\operatorname{grad} \psi, \operatorname{grad} \phi)\} dV. \quad (12)$$

The result holds for any two functions of position ψ and ϕ which within V are finite, continuous, and twice differentiable.

If from (12) we subtract the equation obtained by interchanging ψ and ϕ, we find

$$\iint \left(\psi \frac{\partial\phi}{\partial n} - \phi \frac{\partial\psi}{\partial n}\right) dS = \iiint (\psi \Delta\phi - \phi \Delta\psi) dV. \quad . \quad (13)$$

Both (12) and (13) are referred to as Green's theorem. In electrodynamics, we shall have to make use of them very frequently.*

5. Point Sources.

Hitherto we have always assumed that the sources are distributed continuously, and that the divergence is finite. In point of fact these conditions are fulfilled in all natural vector fields. Cases occur, however, where the distribution of sources approximates to a discontinuous form, the sources becoming condensed practically into points, lines, and surfaces. Since discontinuous distributions are sometimes easier to handle mathematically than continuous ones, we, so to speak, idealize the problem proposed by dealing with a nearly equivalent discontinuous distribution. In doing so, however, if we would make sure of escaping fallacies, we must not lose sight of the fact that we have introduced assumptions not strictly in accord with reality.

* In (12) and (13) note that $\mathbf{n}$ is the normal to S drawn *outwards* from V (or rather, drawn from within V towards S).

In this section we shall discuss the irrotational motion generated by *point* sources. We start from the case of a single point source in a fluid filling the whole of space. From symmetry, the incompressible fluid issuing from the point source spreads out uniformly in all directions. The flow will be radial, and the same quantity of fluid will cross all the spherical surfaces which have the point source for their centre. This defines the strength of the source if, as hitherto, we measure that strength by the volume of fluid issuing from the source per unit time. From now on, however, we shall define the strength of a source in terms of the mass of an ideal fluid whose density, which is still at our disposal, is put equal to $1/4\pi$. This is done in order that the formulæ may bring out clearly the analogy between the field of a moving fluid, and an electric field of force referred to absolute electrostatic units. Thus a source is to have the strength 1, if in one second it generates 4π cubic centimetres of the incompressible fluid. The mass of fluid crossing per second the surface of a sphere of radius r with centre at the source is therefore equal to the strength e:

$$e = \frac{1}{4\pi}\iint v_n\, dS = r^2 v_r; \quad . \quad . \quad . \quad . \quad . \quad (14)$$

conversely the radial velocity is given in terms of the strength by

$$v_r = \frac{e}{r^2};$$

it varies as the inverse square of the distance from the point source. It becomes infinite at the source itself.

The motion is irrotational, the vector $\mathbf{v}$ being expressible as the negative gradient of a potential:

$$\mathbf{v} = -\text{grad}\,\phi, \quad \phi = \frac{e}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Suppose next that we have a series of point sources, h in number, of strengths $e_1, \ldots, e_h$, the fields of which are superimposed; then we can define the resultant field either by geometric addition of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_h$, or more simply by algebraic addition of the scalar potentials $\phi_1, \ldots, \phi_h$:

$$\mathbf{v} = \sum_{i=1}^{h} \mathbf{v}_i = -\text{grad}\,\phi, \quad \phi = \sum_{i=1}^{h} \frac{e_i}{r_i}. \quad . \quad . \quad . \quad (16)$$

For a closed surface, within which there is a number of point sources, the volume of fluid which crosses the surface outwards is equal to 4π times the algebraic sum of the strengths of the sources enclosed by the surface.

If x_i, y_i, z_i are the co-ordinates of the ith source, then the value of the potential ϕ at the point (x, y, z) is, by (16),

$$\phi(x, y, z) = \sum_{i=1}^{h} \frac{e_i}{\sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}}. \qquad (17)$$

It is easily proved that this function, everywhere except at the sources, satisfies Laplace's equation for irrotational motion with no sources,

$$\operatorname{div} \mathbf{v} = -\Delta\phi = -\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) = 0.$$

For a surface S surrounding h point sources $(e_1, \ldots, e_h)$, we have

$$\iint v_n dS = 4\pi(e_1 + e_2 + \ldots + e_h). \quad . \quad . \quad (18)$$

To prove this, apply Gauss's theorem to a region bounded by the surface S and by small spheres $S_1, S_2, \ldots, S_h$ described about the sources as centres. In the region thus marked off, $\operatorname{div}\mathbf{v}$ is everywhere zero, so that

$$\iint_S v_n dS + \iint_{S_1} v_n dS_1 + \ldots + \iint_{S_h} v_n dS_h = 0.$$

Here $\mathbf{n}$ is always to be taken outwards from the space we are dealing with, so that (14) gives

$$\iint v_n dS_1 = -4\pi e_1,$$

and so on. The theorem (18) is therefore proved.

As an application of (17) consider the *potential of a system of sources situated at distances which are all finite, but great in comparison with the distances of the sources from one another.* To find this potential, take the origin of co-ordinates in the neighbourhood of the source system, and expand the expression (17) in powers of x_i, y_i, and z_i, which are now small compared to x, y, z; we therefore put

$$\phi = [\phi_0] + \sum_{i=1}^{i=h}\left(\left[\frac{\partial\phi}{\partial x_i}\right]_0 . x_i + \left[\frac{\partial\phi}{\partial y_i}\right]_0 . y_i + \left[\frac{\partial\phi}{\partial z_i}\right]_0 . z_i + \ldots\right),$$

where the index 0 means that the values of the quantities in question are to be taken in each case for $x_i = y_i = z_i = 0$. Now

$$\frac{\partial}{\partial x_i}\frac{1}{\sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}} = \frac{x-x_i}{(\sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2})^3},$$

so that $$\left[\frac{\partial\phi}{\partial x_i}\right]_0 = \frac{e_i}{r^2}.\frac{x}{r}, \quad \left(r = \sqrt{x^2+y^2+z^2}\right),$$

and we obtain

$$\phi(x, y, z) = \frac{1}{r}\sum_i e_i + \frac{1}{r^2}\left[\frac{x}{r}\sum_i e_i x_i + \frac{y}{r}\sum_i e_i y_i + \frac{z}{r}\sum_i e_i z_i\right] + \ldots .$$

To this order of approximation, the behaviour of our source system is therefore characterized by

(1) its total strength, $e = \sum_i e_i$,

(2) the vector $\mathbf{m} = \sum_i e_i \mathbf{r}_i$ (the components of which are $\Sigma e_i x_i$, $\Sigma e_i y_i$, $\Sigma e_i z_i$), which we call the "moment of the source system".

We have thus

$$\phi = \frac{e}{r} + \frac{(\mathbf{mr})}{r^3} + \ldots,$$

or, if θ is the angle between the vector $\mathbf{m}$ and the radius vector $\mathbf{r}$ to the point (x, y, z),

$$\phi = \frac{e}{r} + \frac{|\mathbf{m}| \,.\, \cos\theta}{r^2} + \ldots \qquad (19)$$

It follows that to a first approximation a system of sources acts at great distances like a point source of strength $\sum_i e_i$.

To bring out the nature of the *second* approximation, we shall consider a system in which $\sum_i e_i$ is zero, and that for the simplest case.

6. Double Sources.

We take two point sources of strengths $+e$ and $-e$. Let $\mathbf{a}$ be the vector represented by the line from the sink $(-e)$ to the source $(+e)$. Then (fig. 2)

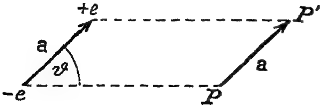

Fig. 2

$$e(\mathbf{r}_+ - \mathbf{r}_-) = e\mathbf{a} = \mathbf{m}$$

is the moment of our system consisting of the two point sources. From this we generate a "*double source*" or "*doublet*" of moment $\mathbf{m}$, by letting $\mathbf{a}$ tend to zero and at the same time e to infinity, in such a way that $e\mathbf{a}$ retains the constant value $\mathbf{m}$.

The point source $+e$ by itself produces at any point P the potential $\phi_+ = e/r$. If we now take the step $\mathbf{a}$ from P to another point P′, we see that, apart from sign, the sink $-e$ produces the same potential

at P as the source $+e$ does at P′. The potential at P due to the two sources together is therefore

$$\phi(\mathrm{P}) = \phi_+(\mathrm{P}) - \phi_+(\mathrm{P}').$$

But, by the definition of a gradient, this is the same as

$$\phi = -(\mathbf{a},\ \mathrm{grad}\,\phi_+) = (-e\mathbf{a},\ \mathrm{grad}\ 1/r).$$

The x-component of grad $1/r$ is

$$\frac{\partial}{\partial x}\frac{1}{\sqrt{(x^2+y^2+z^2)}} = -\frac{1}{r^2}\frac{x}{r}.$$

We accordingly obtain for the potential of the doublet

$$\phi = -(\mathbf{m},\ \mathrm{grad}\ 1/r) = \frac{1}{r^3}(\mathbf{mr}) = \frac{1}{r^2}\,|\,\mathbf{m}\,|\,.\cos\theta, \quad . \quad (20)$$

in agreement with the second approximation in the general formula (19).

In particular, if the doublet is situated at the origin of co-ordinates and $\mathbf{m}$ is in the positive direction of the axis of z, (20) becomes

$$\phi(x, y, z) = \frac{mz}{(x^2+y^2+z^2)^{\frac{3}{2}}}. \quad . \quad . \quad . \quad . \quad (20a)$$

Field gradient and source gradient. If x, y, z are the co-ordinates of a point P in the field, and ξ, η, ζ those of the source s, we must always, when applying the operation "grad" to a function of the distance $r = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$ between the two points, notice carefully whether the differentiation is with respect to the co-ordinates of the source or to those of the point in the field. To prevent errors, it is often useful to indicate explicitly by means of a suffix (f or s) which differentiation is meant. Thus the components in the two cases are:

$$\mathrm{grad}_f\frac{1}{r}:\ \frac{\partial}{\partial x}\frac{1}{r},\ \frac{\partial}{\partial y}\frac{1}{r},\ \frac{\partial}{\partial z}\frac{1}{r};$$

$$\mathrm{grad}_s\frac{1}{r}:\ \frac{\partial}{\partial \xi}\frac{1}{r},\ \frac{\partial}{\partial \eta}\frac{1}{r},\ \frac{\partial}{\partial \zeta}\frac{1}{r}.$$

Obviously we have always $\mathrm{grad}_f\frac{1}{r} = -\mathrm{grad}_s\frac{1}{r}$.

When deducing (20) we differentiated with respect to the co-ordinates of the field point P; to emphasize this, we may now conveniently write

$$\phi = -\left(\mathbf{m},\ \mathrm{grad}_f\frac{1}{r}\right) = \left(\mathbf{m},\ \mathrm{grad}_s\frac{1}{r}\right). \quad . \quad . \quad (20')$$

7. Determination of an Irrotational Vector Field when its Sources are Given.

In § 5 the only motion discussed is that due to point sources. Besides these separate sources of strengths $e_1, e_2, \ldots$, we shall now admit sources which are continuously distributed in space. For this purpose we introduce a function of position $\rho(x, y, z)$ defined by the equation

$$4\pi\rho = \operatorname{div} \mathbf{v}. \quad \ldots \ldots \quad (21)$$

It follows that $\rho\, dV$ is the mass of fluid (of density $1/4\pi$) which leaves the volume element dV per second. To the velocity potential

$$\Sigma \frac{e_i}{r_i}$$

due to the point sources e_i we must add a part representing the effect of the sources $\rho\, dV$ which are distributed throughout the space considered. The aggregate potential may then be conjectured to be

$$\phi = \sum_i \frac{e_i}{r_i} + \iiint \frac{\rho\, dV}{r}, \quad \ldots \ldots \quad (22)$$

or, in more explicit form,

$$\phi(x, y, z) = \sum_i \frac{e_i}{\sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}} + \iiint \frac{\rho(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}}. \quad (22a)$$

We shall now deduce this formula rigorously. First, however, we shall make certain that an irrotational motion is uniquely defined by its sources. To this end we again state the problem whose solution we conjecture to be given by the expression (22*a*).

We wish to find a vector field $\mathbf{v}$ having the following properties:

(*a*) $\mathbf{v}$ is to be irrotational, i.e. there must be a function ϕ such that $\mathbf{v} = -\operatorname{grad} \phi$.

(*b*) $\mathbf{v}$, as well as its potential ϕ, is everywhere finite and continuous, except at certain particular points (point sources). At any point source, however, the difference

$$\phi - \frac{e_i}{r_i}$$

is to be finite and continuous (r_i = distance from the point source), and e_i is then called the strength of the source.

(*c*) Everywhere but at the point sources

$$4\pi\rho = \operatorname{div} \mathbf{v} = -\Delta\phi$$

is to be a prescribed function of position.

(*d*) All sources are to be at a finite distance; in other words, it shall be possible to assign a finite length l so that, outside a sphere of given centre and radius l, there are no point sources and ρ is everywhere zero.

We begin with the proof of uniqueness. If possible, let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two vector fields which satisfy all the conditions (*a*) to (*d*), and which can be derived from the potentials ϕ_1 and ϕ_2 respectively. Apply Green's theorem (12) to the difference

$$\phi_1 - \phi_2 = \chi,$$

replacing both ψ and ϕ in (12) by χ. Thus

$$\iint \chi \frac{\partial \chi}{\partial n} dS = \iiint \{\chi \Delta \chi + (\text{grad } \chi, \text{grad } \chi)\} dV,$$

where we let the boundary tend to infinity. The left side vanishes in the limit, since by condition (*d*) χ tends to 0 like $1/R$ at least, and $\partial\chi/\partial n$ like $1/R^2$ at least, whereas the surface area $\iint dS$ becomes infinite only as R^2. On the right side $\Delta\chi$ is everywhere zero, for $\mathbf{v}_1$ and $\mathbf{v}_2$ have the same sources, so that $\mathbf{v}_1 - \mathbf{v}_2$ is certainly free from sources. We therefore obtain, for the vector $\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2 = \text{grad } \chi$, the result

$$\iiint |\mathbf{w}|^2 dV = 0,$$

which can only be true if the vector $\mathbf{w}$ vanishes everywhere. The two solutions $\mathbf{v}_1$ and $\mathbf{v}_2$ accordingly coincide, i.e. the problem stated can have but one solution at most. *A vector field which is irrotational and free from sources throughout, and which vanishes at infinity, must therefore vanish everywhere.*

To find the solution explicitly, we now apply Green's theorem to our problem, in the form (13):

$$\iint \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}\right) dS = \iiint (\psi \Delta\phi - \phi \Delta\psi) dV,$$

where for ϕ we take the potential function required, and put

$$\psi = \frac{1}{r},$$

r standing for distance from the point P at which the value of ϕ is sought. As the boundary of the integration space we choose: (1) a closed surface tending to infinity, (2) small spherical surfaces round the point sources $e_1, \ldots, e_h$ and the field point P (fig. 3). From the considerations just cited it follows that the external surface in the limit contributes nothing to the left side

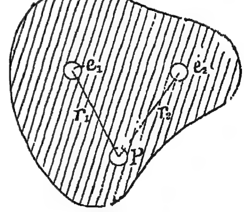

Fig. 3

of (13). The sphere round the first source e_1 contributes

$$\iint\left(\psi\,.\,\frac{e_1}{r^2}-\frac{e_1}{r}\,.\,\frac{\partial\psi}{\partial n}\right)dS_1,$$

for, according to condition (o), ϕ in the neighbourhood of the source may be put in the form $\phi = e_1/r +$ a finite part, and the finite part can contribute nothing when the sphere contracts to a point. The term $(e_1/r)\,.\,\partial\psi/\partial n$ contributes nothing either, for the surface area vanishes as r^2 in the limit. Also ψ may be taken as constant, and equal to $1/r_1$, the reciprocal of the distance of P from the source.

There accordingly remains as the contribution from the first source

$$\psi\,.\,\frac{e_1}{r^2}\,.\,4\pi r^2 = \frac{4\pi e_1}{r_1};$$

and similarly for the other sources. Next, the spherical surface round P contributes

$$\iint\left(\frac{1}{r}\,\frac{\partial\phi}{\partial n}-\phi\,.\,\frac{1}{r^2}\right)dS_{\mathrm{P}}.$$

A calculation similar to that given for the sources shows that in the limit when the volume of the sphere vanishes this becomes

$$-4\pi\phi(\mathrm{P}).$$

On the right side of (13) we are given $\Delta\phi = -4\pi\rho$, and $\Delta\psi$ is zero everywhere. Hence we have

$$\Sigma\;\frac{4\pi e_i}{r_i} - 4\pi\phi(\mathrm{P}) = -\iiint\frac{4\pi\rho}{r}\,dV$$

as the equation giving the potential $\phi(\mathrm{P})$; this agrees exactly with the value conjectured above at equation (22), p. 24.

8. Surface Distributions of Sources. Simple and Double Strata.

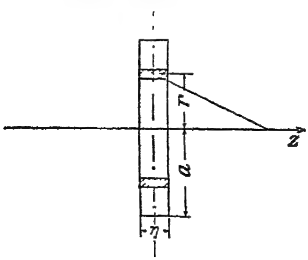

Fig. 4

Up to this point we have made the stipulation that the potential ϕ and the vector $\mathbf{v} = -\mathrm{grad}\ \phi$, apart from individual points (sources), are everywhere to be finite and continuous functions of position. We shall now, as a preliminary to the discussion of surfaces of discontinuity, consider the problem of a circular disk throughout the volume of which there is a uniform distribution of sources, the radius of the disk being a and its thickness η. We confine ourselves to the calculation of the potential on the axis of the disk, which we take as axis of z (fig. 4).

If ρ is the constant strength per unit volume of the disk, then a circular ring of radius r and width dr produces at the point $(0, 0, z)$ the potential

$$\frac{\eta\rho \,.\, 2\pi r\, dr}{\sqrt{r^2 + z^2}}.$$

The potential of the whole disk is therefore

$$\phi(z) = 2\pi\eta\rho \int_0^a \frac{r\, dr}{\sqrt{r^2 + z^2}} = 2\pi\eta\rho\left[\sqrt{a^2 + z^2} - \sqrt{z^2}\right]. \quad (23)$$

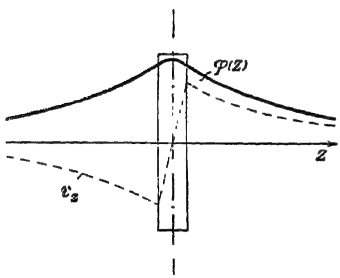

Fig. 5

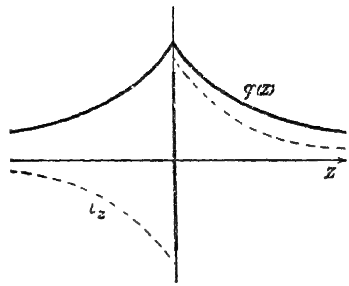

Fig. 5a

Here we have assumed that η is small compared to z. We now pass to the limit $\eta = 0$, $\rho = \infty$, in such a way that the surface density

$$\omega = \eta\rho$$

remains constant. Along with the potential

$$\phi(z) = 2\pi\omega[\sqrt{a^2 + z^2} - \sqrt{z^2}] \quad . \quad . \quad . \quad (23a)$$

we consider also the current velocity $\mathbf{v}$ (v_x and v_y must from symmetry vanish on the z-axis),

$$v_z = -\frac{\partial\phi}{\partial z} = -2\pi\omega\left[\frac{z}{\sqrt{a^2 + z^2}} - \frac{z}{\sqrt{z^2}}\right]. \quad . \quad . \quad (23b)$$

What we are chiefly interested in is the behaviour of our solutions when we pass from one side of the disk to the other (figs. 5, 5a).

For large values of z the potential ϕ has the value $\omega\pi a^2/|z|$. At the disk itself, ϕ is continuous, having on both sides the value $2\pi\omega a$; on the contrary, v_z is positive on the z-positive side, but negative on the z-negative side. For $z = 0$, we obtain, according as we approach $z = 0$ through positive or negative values of z,

$$\mathbf{v}_{+0} = 2\pi\omega, \quad \mathbf{v}_{-0} = -2\pi\omega.$$

Generalizing this result, we are led to the theorem: *When we cross a surface layer of sources (a simple stratum) of strength ω per unit area*

(the surface divergence), the potential ϕ *changes continuously, but the normal component of its gradient,* i.e. $\dfrac{\partial\phi}{\partial n} = -v_n$, *changes abruptly by the amount* $4\pi\omega$.

The proof of the theorem is as follows. Let an arbitrary surface S be given, on which the surface density ω is a given continuous function of position. Suppose that we pass through the surface (in the direction of the normal) at the point P. If we cut out of the surface a small circular disk round P as centre, we can divide the action of the whole surface into two parts; first, the action of the disk—as shown above, this produces a sudden change of amount $4\pi\omega$ in v_n, but no sudden change in ϕ itself; secondly, the action of the rest of the surface—this, being due to sources all at a finite distance from P, cannot produce discontinuities of any kind.

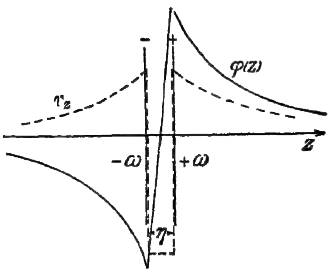

Fig. 6

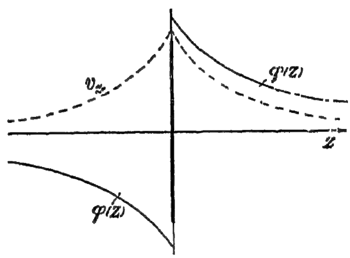

Fig. 6a

We now proceed to consider two parallel circular disks at a distance η from one another, having equal and opposite surface densities $+\omega$ and $-\omega$ (figs. 6, 6*a*). At a point z on the axis, the $+\omega$ disk produces the potential $\phi(z)$ given by (23*a*), the $-\omega$ disk the potential $-\phi(z+\eta)$. Hence the result on the whole (provided η is small relative to z) is

$$\psi = -\frac{\partial\phi}{\partial z}\cdot\eta = 2\pi\omega\eta\left[\frac{z}{\sqrt{z^2}} - \frac{z}{\sqrt{a^2+z^2}}\right],$$

and
$$v_z = -\frac{\partial\psi}{\partial z} = 2\pi\omega\eta\frac{a^2}{\sqrt{(a^2+z^2)^3}}. \quad . \quad . \quad . \quad . \quad (23c)$$

We now, as before, let ω tend to infinity and η to zero, in such a way that the product

$$\tau = \omega\eta$$

retains a constant value. We then call τ the *moment of the double stratum*, in which by this process the two disks coalesce. Our last two formulæ are therefore equivalent to the following general theorem:

When we cross a double stratum of moment τ, the potential ϕ changes suddenly by $4\pi\tau$, but the normal component $\frac{\partial \phi}{\partial n} = -v_n$ undergoes no sudden change.

With the help of these results we can now see at once what effect is produced on an irrotational fluid motion by assigned sudden changes of ϕ and v_n at a given surface S.

We distinguish the two sides of the given surface by suffixes 1 and 2; $\mathbf{n}_1$ and $\mathbf{n}_2$ are the outward normals of the regions bounded by the surface, or, in other words, the normals drawn from the interiors towards the surface (cf. fig. 7).

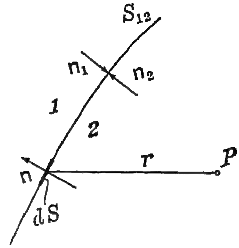

Fig. 7.

The discontinuity in the normal component of $\mathbf{v}$ is then $v_{n_1} + v_{n_2}$. Let now the discontinuities of v_n and ϕ be assigned as functions of position on the bounding surface S or, as we may write it, S_{12}:

$$\left.\begin{aligned} 4\pi\omega &= -(v_{n_1} + v_{n_2}) \\ 4\pi\tau &= \phi_1 - \phi_2. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

From the results already found we might anticipate that an element dS_{12} of the surface acts on a point P(x, y, z) outside the surface (1) like a source of strength ωdS_{12} on account of the discontinuity in v_n, (2) like a double stratum of moment $\tau\mathbf{n}dS_{12}$ on account of the discontinuity in ϕ. Here $\mathbf{n}$ is that normal unit vector which points from the lower to the higher potential (the direction $\mathbf{n}$ in fig. 7 corresponds to the case $\phi_1 > \phi_2$). We therefore expect, from (15) and (20), that the contribution of dS_{12} to the potential at a point P is (cf. p. 23)

$$\frac{\omega\, dS_{12}}{r} - \left(\tau\mathbf{n}\, dS_{12} \cdot \operatorname{grad}_f \frac{1}{r}\right),$$

and therefore that the total potential due to our surface of discontinuity is

$$\phi = \iint \frac{\omega}{r}\, dS_{12} - \iint \tau\left(\mathbf{n}\operatorname{grad}_f \frac{1}{r}\right) dS_{12}. \quad . \quad . \quad (25)$$

We shall now verify this formula by means of Green's theorem

$$\iint \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) dS = \iiint (\phi \Delta\psi - \psi\Delta\phi)\, dV.$$

We again denote by r the distance from the field point P, and put

$$\psi = \frac{1}{r}.$$

Let ϕ be the required potential possessing the properties of discontinuity at the surface S_{12} which are assigned by (24). Everywhere else we are to have $\Delta\phi = 0$.

As the boundary of the integration space we now take: (1) an external surface, which (as in § 7, p. 25) we send to infinity and which contributes nothing to the surface integral; (2) a sphere round the field point P (the surface S_3 in fig. 8) which, as shown in § 7, contributes to the surface integral

$$\phi(\mathrm{P}) \cdot \frac{1}{r^2} \cdot 4\pi r^2 = 4\pi\phi(\mathrm{P});$$

(3) a surface or surfaces S_1, S_2 enveloping S_{12} and excluding it from the integration space. In the region thus bounded, ϕ and ψ are everywhere finite and continuous, and satisfy the equations $\Delta\phi = 0$, $\Delta\psi = 0$, so that the right-hand side in Green's equation is zero. There therefore remains only

$$4\pi\phi(\mathrm{P}) = \iint_{s_1}\left(\psi\frac{\partial\phi}{\partial n_1} - \phi\frac{\partial\psi}{\partial n_1}\right)dS_1 + \iint_{s_2}\left(\psi\frac{\partial\phi}{\partial n_2} - \phi\frac{\partial\psi}{\partial n_2}\right)dS_2.$$

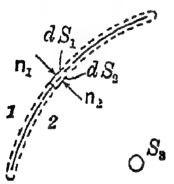

Fig. 8

Next, on the right side, take together each pair of elements dS_1 and dS_2 which face each other (fig. 8). Taking the normal direction $\mathbf{n}$ of the given surface to coincide with $\mathbf{n}_2$, we have $\mathbf{n}_1 = -\mathbf{n}$, $\mathbf{n}_2 = \mathbf{n}$. Thus

$$\frac{\partial\psi}{\partial n_1} = -\frac{\partial}{\partial n}\left(\frac{1}{r}\right); \quad \frac{\partial\psi}{\partial n_2} = \frac{\partial}{\partial n}\left(\frac{1}{r}\right).$$

On taking S_1 and S_2 together, we obtain an integral over the surface of discontinuity:

$$4\pi\phi(\mathrm{P}) = \iint\psi\left(\frac{\partial\phi}{\partial n_1} + \frac{\partial\phi}{\partial n_2}\right)dS_{12} + \iint\frac{\partial\psi}{\partial n}(\phi_1 - \phi_2)dS_{12}.$$

If in this we put

$$\psi = \frac{1}{r}; \quad \frac{\partial\psi}{\partial n} = \left(\mathbf{n}, \operatorname{grad}_s\frac{1}{r}\right) = -\left(\mathbf{n}, \operatorname{grad}_f\frac{1}{r}\right),$$

and also introduce, in place of $\dfrac{\partial\phi}{\partial n_1} + \dfrac{\partial\phi}{\partial n_2}$ and of $\phi_1 - \phi_2$, the discontinuities assigned by (24), we obtain exactly formula (25).

9. The Uniform Double Stratum.

A double stratum of moment τ produces according to (25) the potential

$$\phi = \iint\tau\left(\mathbf{n}\operatorname{grad}_s\frac{1}{r}\right)dS_{12}.$$

This expression admits of an important transformation. If $\mathbf{r}$ is the vector drawn from the field point to the element dS_{12}, then

$$\operatorname{grad}_s \frac{1}{r} = -\frac{\mathbf{r}}{r^3},$$

so that
$$\left(\mathbf{n}\,.\operatorname{grad}_s \frac{1}{r}\right) = -\frac{(\mathbf{nr})}{r^3} = -\frac{\cos(\mathbf{n}, \mathbf{r})}{r^2}.$$

The potential therefore becomes

$$\phi = -\iint \tau \frac{\cos(\mathbf{n}, \mathbf{r})\, dS_{12}}{r^2}.$$

Now
$$\pm \frac{\cos(\mathbf{n}, \mathbf{r})\, dS_{12}}{r^2} = d\Omega$$

is the *solid angle* under which the surface element dS_{12} would be seen by an observer at the field point. We thus obtain

$$\phi = \iint \tau\, d\Omega; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where $d\Omega$ is to be taken positive or negative according as the observer at the field point has the positive or negative side of the surface element next to him (fig. 9).

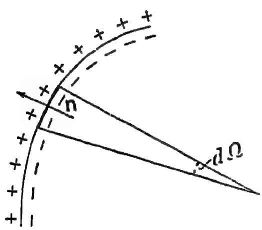

Fig. 9

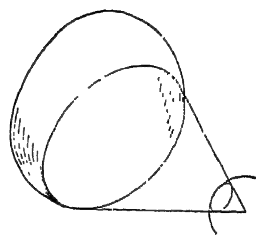

Fig. 10

When τ is constant over the surface, the double stratum is said to be uniform. In this case (26) assumes the simple form

$$\phi = \tau\Omega.$$

The potential of a uniform double stratum of moment τ is equal to the product of τ and the solid angle Ω under which the curve bounding the double stratum is seen from the field point (fig. 10).

If we contract the boundary curve to a point, we obtain a closed double stratum. For this, Ω becomes zero at a point outside the surface, but 2π for a point within it. Hence if a uniform double stratum is closed (with its positive side outwards), $\phi = 0$ at all exterior points, but $\phi = -4\pi\tau$ at all interior points.

The current $\mathbf{v} = -\text{grad}\ \phi$ produced by a uniform double stratum is found from the following consideration. If $\delta\phi$ is the change in ϕ for a displacement $\delta\mathbf{a}$, then $(\mathbf{v}\,.\,\delta\mathbf{a}) = -\delta\phi = -\tau\,\delta\Omega$. Here $\delta\Omega$ is the change in the solid angle due to the displacement of the field point by $\delta\mathbf{a}$. Clearly we would obtain the same change $\delta\Omega$ if we kept the field point fixed and displaced the double stratum by $-\delta\mathbf{a}$. In this displacement a line element $d\mathbf{s}$ of the boundary curve sweeps out the element of area $-[\delta\mathbf{a}\,.\,d\mathbf{s}]$, which is seen from the field point under the solid angle

$$-\frac{(\mathbf{r}[\delta\mathbf{a}\,.\,d\mathbf{s}])}{r^3} = -\frac{([d\mathbf{s}\,.\,\mathbf{r}]\,\delta\mathbf{a})}{r^3}.$$

From this $\delta\Omega$ is obtained by integration over the boundary curve. Thus

$$(\mathbf{v}\,.\,\delta\mathbf{a}) = \left(\delta\mathbf{a}\,.\,\tau\oint\frac{[d\mathbf{s}\,.\,\mathbf{r}]}{r^3}\right)$$

for every direction $\delta\mathbf{a}$; therefore

$$\mathbf{v} = \tau\oint\frac{[d\mathbf{s}\,.\,\mathbf{r}]}{r^3}. \qquad \text{(26}a\text{)}$$

10. Curl, and Stokes's Theorem.

At p. 15 we called a vector field irrotational, if for every closed path the line integral

$$\oint v_s ds \qquad (27)$$

vanishes. The necessary and sufficient condition for this was found to be that $\mathbf{v}$ must be expressible in the form $(-\text{grad}\ \phi)$, i.e. that a scalar ϕ must exist such that

$$-\frac{\partial\phi}{\partial x} = v_x, \quad -\frac{\partial\phi}{\partial y} = v_y, \quad -\frac{\partial\phi}{\partial z} = v_z.$$

But from these relations it follows at once that for an irrotational vector $\mathbf{v}$ the three quantities

$$\left.\begin{aligned} w_x &= \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ w_y &= \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ w_z &= \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{aligned}\right\} \qquad (28)$$

must be everywhere zero. This suggests that the three quantities w_x, w_y, w_z should be taken as a measure of the intensity of rotation in the vector field.

We shall eventually (see p. 35, at equation (30)) prove that the quantities w_x, w_y, w_z defined in (28) represent the components of an actual vector (cf. p. 6), which we shall denote by the symbol curl **v**. We therefore (in the first instance purely formally) write equation (28) in the form

$$\mathbf{w} = \operatorname{curl} \mathbf{v}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28a)$$

or

$$\operatorname{curl} \mathbf{v} = \mathbf{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right). \quad (28b)$$

In order to bring out the connexion between the line integral (27) and the new quantity **w**, we begin with the case (fig. 11) of an area in the plane of xy, round which we make a circuit in the positive sense, i.e. in such a sense that the circuit has the right-handed screw relation to advance along the positive direction of the z-axis. For this circuit we calculate the line integral

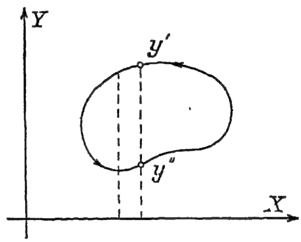

Fig. 11

$$\oint v_s ds = \oint (v_x dx + v_y dy).$$

In calculating the first term $\int v_x dx$, we take together the two elements of the curve corresponding to the same dx (for y' and y'', where $y' > y''$). Here dx is positive for the smaller value y'' of y. The contribution from these two elements to $\int v_x dx$ is therefore

$$-dx\,\{v_x(y') - v_x(y'')\} = -dx\,.\int_{y''}^{y'} \frac{\partial v_x}{dy}\,\partial y.$$

Hence altogether

$$\oint v_x dx = -\int\int \frac{\partial v_x}{\partial y}\, dx\, dy,$$

the integral on the right being taken over the whole area. Treating the term $\int v_y dy$ in a similar way, we therefore find

$$\oint v_s ds = \int\int_S \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) dx\, dy.$$

The integrand on the right is precisely the quantity w_z introduced in (28). If we choose S so small that w_z may be regarded as constant within S, then

$$\oint v_s ds = w_z S,$$

or—somewhat more exactly—

$$w_z = \lim_{S=0} \frac{1}{S} \oint v_s ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Thus in order to define the quantity w_z at any point of the field we can either calculate the derivatives appearing in (28) or—and this is a more suggestive method—we can form the integral $\oint v_s ds$ round an area S including the point in question, so that the right-handed screw normal is parallel to Oz, and divide the integral by the area; w_z is then, by (29), the limit to which the quotient tends for S tending to zero. Similarly we can of course obtain w_x and w_y by taking the directed normals to the chosen areas parallel to Ox and Oy.

We now inquire: how may the quantity

$$\lim_{S=0} \frac{1}{S} \oint v_s ds$$

be represented if we take the normal $\mathbf{n}$ to S in any arbitrary direction, say with direction cosines $\cos(\mathbf{n}, x)$, $\cos(\mathbf{n}, y)$, $\cos(\mathbf{n}, z)$? Take the origin of co-ordinates near the element of area, so that in a region containing the element S we can represent the vector $\mathbf{v}$ by the first few terms of a Taylor expansion,

$$v_x = v_{x_0} + \left(\frac{\partial v_x}{\partial x}\right)_0 x + \left(\frac{\partial v_x}{\partial y}\right)_0 y + \left(\frac{\partial v_x}{\partial z}\right)_0 z,$$

with corresponding expressions for v_y and v_z. When we insert these values in

$$\oint v_s ds = \oint (v_x dx + v_y dy + v_z dz),$$

the terms in v_{x_0} and $(\partial v_x/\partial x)_0$ vanish, since the integrals

$$v_{x_0} \oint dx \quad \text{and} \quad \left(\frac{\partial v_x}{\partial x}\right)_0 \oint x dx = \left(\frac{\partial v_x}{\partial x}\right)_0 \frac{1}{2} \oint d(x^2)$$

are clearly zero. There only remains then

$$\begin{aligned}\oint v_s ds = {} & \left(\frac{\partial v_x}{\partial y}\right)_0 \oint y dx + \left(\frac{\partial v_x}{\partial z}\right)_0 \oint z dx \\ & + \left(\frac{\partial v_y}{\partial x}\right)_0 \oint x dy + \left(\frac{\partial v_y}{\partial z}\right)_0 \oint z dy \\ & + \left(\frac{\partial v_z}{\partial x}\right)_0 \oint x dz + \left(\frac{\partial v_z}{\partial y}\right)_0 \oint y dz.\end{aligned}$$

But the integrals which appear on the right are simply the projections of the given area S on the three co-ordinate planes. In fact, taking

account of the algebraic signs of the integrals as depending on the currencies of the paths, we have

$$S\cos(\mathbf{n}, z) = \oint x\,dy = -\oint y\,dx,$$
$$S\cos(\mathbf{n}, y) = \oint z\,dx = -\oint x\,dz,$$
$$S\cos(\mathbf{n}, x) = \oint y\,dz = -\oint z\,dy.$$

The result is therefore

$$\left.\begin{aligned}\frac{1}{S}\oint v_s ds = \cos(\mathbf{n}, x)\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\\ + \cos(\mathbf{n}, y)\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\\ + \cos(\mathbf{n}, z)\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Until we had proved this equation we had no right to call the quantities w_x, w_y, w_z introduced in (28) the components of a vector. But we can now do so, for (30) states that for a small area S, oriented in any way and having the right-handed screw normal $\mathbf{n}$, we have

$$\oint v_s ds = S\,.\,w_n = S\,.\,|\,w\,|\,.\cos(\mathbf{n}, \mathbf{w}). \quad . \quad . \quad (30a)$$

In words: "the line integral $\oint v_s ds$ is equal to the product of the area enclosed and the component of the vector curl $\mathbf{v}$ in the direction of the normal to that area ".

This theorem contains at the same time a definition of curl $\mathbf{v}$ which is independent of co-ordinates.

Stokes's Theorem.—Equations (29) to (30a) hold rigorously only in the limit $S = 0$. With their help, however, we can at once deduce a general theorem for a line integral taken along an arbitrary closed curve. For this purpose consider a surface bounded by the given curve s, but otherwise quite arbitrary. By the currency assigned to s, every element of this surface has associated with it a normal direction definitely fixed by the right-handed screw rule. We divide up this surface into elements of area dS_1, dS_2, dS_3, &c., all small (fig. 12). If for all such elements separately we now form the integrals $\int v_s ds$, and add these elementary contour integrals, the contributions arising from the common boundary of any two elements (e.g. dS_1 and dS_2) exactly cancel each other, since they

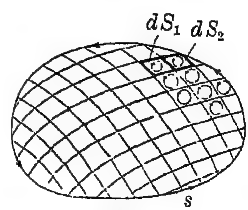

Fig. 12

always occur in pairs, and that with opposite signs. Thus after addition there is left only the integral over the original bounding contour

$$\oint_S v_s ds = \oint_{dS_1} v_s ds + \oint_{dS_2} v_s ds + \dots .$$

We can now apply our equation (30*a*) to every separate surface element. We thus obtain Stokes's theorem

$$\oint v_s ds = \iint w_n dS; \quad \mathbf{w} = \text{curl } \mathbf{v}. \quad . \quad . \quad . \quad (31)$$

It is to be noted that the surface S was taken quite arbitrarily through the given space curve. If then we take two different surfaces S_1 and S_2 through the given curve, we must by (31) have

$$\iint w_n dS_1 = \iint w_n dS_2.$$

But the two surfaces together define a region of space, of which they form the complete boundary. If in the last equation we reverse the direction of the normal to *one* of the surfaces (say S_2), then

$$\iint w_n dS_1 + \iint w_n dS_2 = 0,$$

and this is the total flux of the vector $\mathbf{v}$ out of the space bounded by the two surfaces together (fig. 13). We see therefore that *the vector* $\mathbf{w} =$ *curl* $\mathbf{v}$ *is always solenoidal,* i.e. the relation

$$\text{div curl } \mathbf{v} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

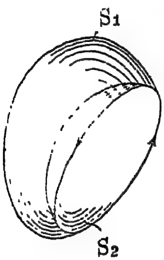

Fig. 13

holds in all cases; as may also be proved at once from (28).

We shall now calculate the vector curl curl $\mathbf{v}$, of which use will be made later. Its x-component is obviously

$$\frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) - \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

so that in vector notation we have

$$\text{curl curl } \mathbf{v} = \text{grad div } \mathbf{v} - \Delta \mathbf{v}. \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Further, we have of course always

$$\text{curl grad } \phi = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

We shall also use later the relation

$$\text{div } [\mathbf{AB}] = (\mathbf{B} \text{ curl } \mathbf{A}) - (\mathbf{A} \text{ curl } \mathbf{B}), \quad . \quad . \quad . \quad (35)$$

which is true for any two vectors $\mathbf{A}$ and $\mathbf{B}$, as may easily be verified by writing it out in terms of co-ordinates.

11. Calculation of a Vector Field from its Sources and Vortices.

We saw in § 7 (p. 24) how an irrotational vector field can be calculated by a general process when its sources are given. In this section we shall deal with the general problem of calculating a vector field whose sources and vortices * are prescribed. As before, we assume that all sources and vortices are at a finite distance. The problem therefore is, to define a vector field $\mathbf{v}$ so that, simultaneously,

$$\operatorname{div} \mathbf{v} = 4\pi\rho, \quad \text{(36a)}$$

$$\operatorname{curl} \mathbf{v} = 4\pi\mathbf{c}, \quad \text{(36b)}$$

the scalar ρ and the vector $\mathbf{c}$ being given for every point of space. Here $\mathbf{c}$ cannot be assigned quite arbitrarily; in fact, in view of (32) we must have

$$\operatorname{div} \mathbf{c} = 0 \quad \text{(36c)}$$

everywhere. That at most only one solution can exist of equations (36*a*, *b*) follows from the theorem proved in § 7 (p. 25) to the effect that a vector field which is everywhere irrotational and solenoidal must vanish. For the difference of two solutions must necessarily satisfy the equations $\operatorname{div} \mathbf{v} = 0$ and $\operatorname{curl} \mathbf{v} = 0$ throughout the field.

We proceed to investigate the solution of our problem. For this purpose we split up the vector $\mathbf{v}$ of which we are in search into two parts $\mathbf{v}_1$ and $\mathbf{v}_2$,

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2, \quad \text{(36d)}$$

and try to satisfy (36*a*, *b*) by assuming

$$\operatorname{curl} \mathbf{v}_1 = 0, \qquad \operatorname{div} \mathbf{v}_1 = 4\pi\rho; \quad \text{(36e)}$$

$$\operatorname{curl} \mathbf{v}_2 = 4\pi\mathbf{c}, \qquad \operatorname{div} \mathbf{v}_2 = 0. \quad \text{(36f)}$$

We thus split up the required vector field $\mathbf{v}$ *into an irrotational field* $\mathbf{v}_1$ *having the prescribed sources, and a solenoidal field* $\mathbf{v}_2$ *having the prescribed vortices.*

This splitting up can certainly only be done in one way. For the irrotational part $\mathbf{v}_1$ is, by § 7, uniquely defined by the condition (36*e*). We can even give its value explicitly; $\mathbf{v}_1$ is derivable from a potential ϕ, with

$$\mathbf{v}_1 = -\operatorname{grad} \phi, \text{ where } \phi = \iiint \frac{\rho\, dV}{r}. \quad \text{(36g)}$$

We are now left only with the problem of defining the solenoidal field $\mathbf{v}_2$ in accordance with (36*f*). But the solenoidal property allows us to express $\mathbf{v}_2$ as the curl of another vector $\mathbf{A}$,

$$\mathbf{v}_2 = \operatorname{curl} \mathbf{A}. \quad \text{(36h)}$$

* Ger., *Quellen und Wirbeln.*

The vector $\mathbf{A}$ so defined is called the *vector potential* of $\mathbf{v}_2$. And just as the first equation (36*g*) only defines ϕ within an arbitrary constant, so we can add to the vector potential $\mathbf{A}$ an arbitrary irrotational vector, without (36*h*) being thereby altered. We shall take advantage of this part of $\mathbf{A}$ which we can still choose arbitrarily, to subject $\mathbf{A}$ to the restrictive condition,

$$\operatorname{div} \mathbf{A} = 0. \qquad (36i)$$

In terms of $\mathbf{A}$ (36*f*) becomes

$$\operatorname{curl} \operatorname{curl} \mathbf{A} = 4\pi \mathbf{c}.$$

Hence by formula (33), taking account of (36*i*), we have

$$\Delta \mathbf{A} = -4\pi \mathbf{c}, \qquad (36k)$$

an equation exactly analogous to Laplace's equation for the scalar potential

$$\Delta \phi = -4\pi \rho,$$

the solution of which has already been given in (36*g*). The analogy enables us to write down the solution of (36*k*) at once. It is

$$\mathbf{A} = \iiint \frac{\mathbf{c}\,dV}{r}. \qquad (36l)$$

We have accordingly solved the problem with which we started. The result runs

$$\mathbf{v} = -\operatorname{grad} \phi + \operatorname{curl} \mathbf{A},$$

$$\phi = \iiint \frac{\rho\,dV}{r}, \quad \mathbf{A} = \iiint \frac{\mathbf{c}\,dV}{r}.$$

It only remains to assure ourselves that the vector field given by (36*l*) is actually solenoidal, as is required by (36*i*). Now in the first place (36*l*) obviously gives (cf. p. 23)

$$\operatorname{div} \mathbf{A} = \iiint \left(\mathbf{c} \operatorname{grad}_r \frac{1}{r}\right) dV = -\iiint \left(\mathbf{c} \operatorname{grad}_s \frac{1}{r}\right) dV.$$

Also,
$$\left(\mathbf{c} \operatorname{grad}_s \frac{1}{r}\right) = \operatorname{div} \left(\frac{\mathbf{c}}{r}\right) - \frac{1}{r} \operatorname{div} \mathbf{c}.$$

But by (36*c*) $\mathbf{c}$ must always be given as a solenoidal vector, so that we find

$$\operatorname{div} \mathbf{A} = -\iiint \operatorname{div} \left(\frac{\mathbf{c}}{r}\right) dV = -\iint \frac{c_n}{r}\, dS.$$

Now the whole vortex system is interior to S; consequently $c_n = 0$ at every point of S. The vector $\mathbf{A}$ is therefore actually solenoidal.

If in equation (10), p. 19, we take grad $\psi = \mathbf{v}_1$, $\mathbf{A} = \mathbf{v}_2$, and apply the equation to the whole field, the surface integral vanishes in the limit when the surface is pushed to infinity. Hence

$$\iiint (\mathbf{v}_1 \cdot \mathbf{v}_2) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (36m)$$

We have therefore the theorem:

The volume integral, taken over the whole system, of the scalar product of an irrotational and a solenoidal vector is always zero.

12. Time Rate of Change of the Flux through a Moving Element of Area.

Let $\mathbf{A}$ be an arbitrary velocity field, which may vary with the time, so that $\mathbf{A}$ is any (continuous and differentiable) function of x, y, z and t. Then $\iint A_n dS$ is the flux through or across a surface S, i.e. the volume of fluid passing across S per unit time. If the surface is at rest, the time rate of change of the flux is

$$\iint \dot{A}_n dS = \iint \frac{\partial A_n}{\partial t} dS.$$

But if the surface S itself is moving, the mere change of the position of S in the field $\mathbf{A}$ will cause changes in the flux. We now *define* a new kind of differentiation with respect to the time by the symbol $\underline{\dot{\mathbf{A}}}$ as follows:

$$\frac{d}{dt}\iint A_n dS = \iint \underline{\dot{A}}_n dS. \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$\underline{\dot{\mathbf{A}}}$ is therefore a vector the flux of which across the moving surface is equal to the time rate of increase of the flux of $\mathbf{A}$ across the same surface. In order to calculate $\underline{\dot{\mathbf{A}}}$, we must of course know the motion of our surface exactly. Let this motion be described by the vector $\mathbf{u}$, which we suppose to be given for every element dS of the surface and to represent the velocity of the element.

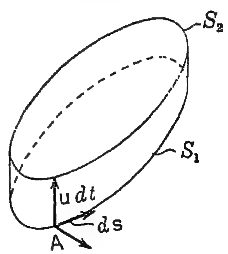

Fig. 14

Now let S_1 (fig. 14) be the position of our surface S at time $t - dt$, and S_2 its position at time t. We obtain S_2 from S_1 by giving every surface element of S_1 the displacement $\mathbf{u}\,dt$. The surfaces S_1 and S_2, along with the small strip traced out in the motion by the curve which bounds them, include the volume $dt \,.\iint u_n dS$.

The time rate of change of the flux of **A** across S is now to be calculated from the difference between the flux across S_2 at time t and the flux across S_1 at time $t - dt$:

$$\frac{d}{dt}\iint A_n dS = \frac{\iint A_{n,t}\, dS_2 - \iint A_{n,t-dt}\, dS_1}{dt}.$$

We apply Gauss's theorem to the flat box-shaped space bounded by S_1, S_2 and the strip connecting their edges, and that for the time t. Here the normal to S_2 is the outward normal, that to S_1 the inward one. Also, a surface element of the side face, as regards both numerical value and outward normal direction, is given by $[d\mathbf{s}\,\mathbf{u}]dt$. Hence Gauss's theorem gives

$$\iint_{S_2} A_{n,t}\, dS_2 + dt \oint (\mathbf{A}[d\mathbf{s}\mathbf{u}]) - \iint_{S_1} A_{n,t}\, dS_1$$
$$= dt \iint (\operatorname{div} \mathbf{A})\, u_n\, dS.$$

We have further

$$\iint A_{n,t-dt}\, dS_1 = \iint A_{n,t}\, dS_1 - \iint \frac{\partial A_n}{\partial t}\, dS_1\, dt$$

Hence, by subtraction,

$$\iint A_{n,t}\, dS_2 - \iint A_{n,t-dt}\, dS_1$$
$$= dt\left\{\iint \dot{A}_n\, dS_1 + \iint (\operatorname{div} \mathbf{A}) u_n\, dS_1 - \oint (\mathbf{A}[d\mathbf{s}\mathbf{u}])\right\}.$$

The last term on the right can be transformed by Stokes's theorem: thus

$$\oint \mathbf{A}[d\mathbf{s}\mathbf{u}] = \oint ([\mathbf{u}\mathbf{A}]d\mathbf{s}) = \iint (\operatorname{curl}\, [\mathbf{u}\mathbf{A}])_n\, dS;$$

so that we obtain finally

$$\frac{d}{dt}\iint A_n dS = \iint \{\dot{A}_n + u_n \operatorname{div} \mathbf{A} - (\operatorname{curl}\, [\mathbf{u}\mathbf{A}])_n\}\, dS.$$

This also determines the vector $\underline{\dot{\mathbf{A}}}$ defined by (37), viz. we have

$$\underline{\dot{\mathbf{A}}} = \dot{\mathbf{A}} + \mathbf{u} \operatorname{div} \mathbf{A} - \operatorname{curl}\, [\mathbf{u}\mathbf{A}]. \quad . \quad . \quad . \quad (37a)$$

In electrodynamics this expression is particularly important in the case when we have to calculate the rate of change of the flux of induction through a moving coil.

13. Orthogonal Curvilinear Co-ordinates.

Many calculations in electrodynamics can be simplified by choosing, instead of a Cartesian co-ordinate system, another kind of system which takes advantage of the relations of symmetry involved in the particular problem under consideration. Let the new co-ordinates

u_1, u_2, u_3 be defined by specifying the Cartesian co-ordinates x, y, z as functions of u_1, u_2, u_3:

$$x = x(u_1, u_2, u_3),$$
$$y = y(u_1, u_2, u_3),$$
$$z = z(u_1, u_2, u_3).$$

We confine ourselves to the case when the three families of surfaces $u_1 = \text{const.}$, $u_2 = \text{const.}$, $u_3 = \text{const.}$, are orthogonal to one another. In that case the line element $ds = \sqrt{(dx^2 + dy^2 + dz^2)}$ is given by the expression

$$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2, \quad . \quad . \quad . \quad (38)$$

where h_1, h_2, h_3 may be functions of u_1, u_2, u_3. Also, we adopt the convention that the new co-ordinate system shall be a right-handed system, like the old.

Consider the infinitesimal parallelepiped, whose diagonal is the line element ds, and whose faces coincide with the planes u_1, or u_2, or $u_3 = \text{const.}$ The lengths of its edges are then (fig. 15) h_1du_1, h_2du_2, h_3du_3, and its volume is $h_1h_2h_3du_1du_2du_3$. Further, let $\phi(u_1, u_2, u_3)$ be a scalar function, and **A** a vector field with the components A_1, A_2, A_3 in the three directions in which u_1, u_2, u_3 increase.

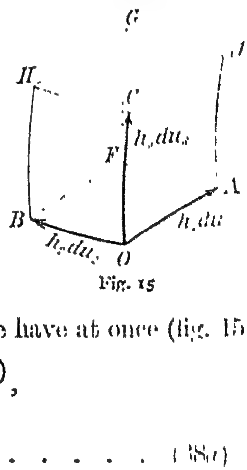

Fig. 15

For the u_1-component of the *gradient* of ϕ we have at once (fig. 15)

$$(\text{grad}\,\phi)_1 = \lim_{du_1=0} \frac{\phi(A) - \phi(O)}{h_1du_1},$$

or

$$(\text{grad}\,\phi)_1 = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38a)$$

and similarly for the directions 2 and 3.

To calculate the *divergence* of a vector **A** we go back to Gauss's theorem. The flux through the area OBHC, taken in the direction of the *outward* normal, is $-A_1h_2h_3du_2du_3$, while the flux through AFGJ is

$$A_1h_2h_3du_2du_3 + \frac{\partial}{\partial u_1}(A_1h_2h_3)du_1du_2du_3.$$

From these and the corresponding expressions for the other two pairs of surfaces, using the result

$$\text{div}\,\mathbf{A}\,.\,h_1h_2h_3du_1du_2du_3 = \iint A_n dS,$$

we obtain the equation

$$\text{div}\,\mathbf{A} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right\}. \quad (38b)$$

The component 1 of the *curl* is found by applying Stokes's theorem to the surface OBHC. Here we have

$$\int_B^H A_s ds + \int_C^O A_s ds = \frac{\partial}{\partial u_2}(A_3 h_3 du_3) du_2, \text{ \&c.,}$$

so that
$$(\text{curl } \mathbf{A})_1 = \frac{1}{h_2 h_3}\left\{\frac{\partial}{\partial u_2}(A_3 h_3) - \frac{\partial}{\partial u_3}(A_2 h_2)\right\}, \quad . \quad (38c)$$

and similarly, by cyclic changes of the indices, for the directions 2 and 3.

Finally, *Laplace's operator* $\Delta =$ div grad is obtained by combining (38*a*) and (38*b*):

$$\Delta\phi = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right\}. \quad (38d)$$

We shall now apply these general formulæ to two special cases which are particularly important in applications.

(*a*) *Cylindrical Co-ordinates.*

$$\begin{aligned} x &= r\cos\alpha \\ y &= r\sin\alpha \\ z &= z \\ ds^2 &= dr^2 + r^2 d\alpha^2 + dz^2. \end{aligned}$$

We have therefore in this case

$$\begin{matrix} u_1 = r & & h_1 = 1 \\ u_2 = \alpha & \text{and} & h_2 = r \\ u_3 = z & & h_3 = 1. \end{matrix}$$

Hence, by equations (38*a*) to (38*d*),

$$\left.\begin{aligned} \text{grad}_r\,\phi &= \frac{\partial\phi}{\partial r},\ \text{grad}_\alpha\,\phi = \frac{1}{r}\frac{\partial\phi}{\partial\alpha},\ \text{grad}_z\,\phi = \frac{\partial\phi}{\partial z}, \\ \text{div}\,\mathbf{A} &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\alpha}{\partial\alpha} + \frac{\partial A_z}{\partial z}, \\ (\text{curl }\mathbf{A})_r &= \frac{1}{r}\frac{\partial A_z}{\partial\alpha} - \frac{\partial A_\alpha}{\partial z}, \\ (\text{curl }\mathbf{A})_\alpha &= \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \\ (\text{curl }\mathbf{A})_z &= \frac{1}{r}\left\{\frac{\partial}{\partial r}(rA_\alpha) - \frac{\partial A_r}{\partial\alpha}\right\}, \\ \text{and}\quad \Delta\phi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\alpha^2} + \frac{\partial^2\phi}{\partial z^2}. \end{aligned}\right\} \quad (38e)$$

(*b*) *Spherical Polar Co-ordinates.*

$$
\begin{aligned}
x &= r\sin\theta\cos\alpha \\
y &= r\sin\theta\sin\alpha \\
z &= r\cos\theta \\
ds^2 &= dr^2 + r^2\sin^2\theta\, d\alpha^2 + r^2 d\theta^2.
\end{aligned}
$$

We have therefore in (38) to (38*d*) to put

$$
\begin{aligned}
u_1 &= r & & & h_1 &= 1 \\
u_2 &= \theta & &\text{and} & h_2 &= r \\
u_3 &= \alpha & & & h_3 &= r\sin\theta.
\end{aligned}
$$

Then

$$
\left.
\begin{aligned}
\operatorname{grad}_r\phi &= \frac{\partial\phi}{\partial r},\quad \operatorname{grad}_\theta\phi = \frac{1}{r}\frac{\partial\phi}{\partial\theta},\quad \operatorname{grad}_\alpha\phi = \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\alpha}, \\
\operatorname{div}\mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\alpha}{\partial\alpha}, \\
(\operatorname{curl}\mathbf{A})_r &= \frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta A_\alpha) - \frac{\partial A_\theta}{\partial\alpha}\right\}, \\
(\operatorname{curl}\mathbf{A})_\theta &= \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\alpha} - \frac{1}{r}\frac{\partial(rA_\alpha)}{\partial r}, \\
(\operatorname{curl}\mathbf{A})_\alpha &= \frac{1}{r}\left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right\}, \\
\Delta\phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\alpha^2}.
\end{aligned}
\right\} \quad (38f)
$$

14. Tensors. Polar and Axial Vectors.

As we saw at equation (9), p. 6, a vector $\mathbf{a}$ gives us the means of associating any direction $\mathbf{s}$ in space with a scalar

$$a_s = a_x\cos(\mathbf{s}, x) + a_y\cos(\mathbf{s}, y) + a_z\cos(\mathbf{s}, z),$$

the component of the vector in that direction; a_s is a homogeneous linear function both of the vector components a_x, a_y, a_z and of the direction cosines of $\mathbf{s}$.

Now it often happens in physics that a *vector* $\mathfrak{q}$ is associated with a direction by the above law:

$$\mathfrak{q} = \mathfrak{q}_1\cos(\mathbf{s}, x) + \mathfrak{q}_2\cos(\mathbf{s}, y) + \mathfrak{q}_3\cos(\mathbf{s}, z); \quad . \quad . \quad (39)$$

where we have three *vectors* $\mathfrak{q}_1$, $\mathfrak{q}_2$, $\mathfrak{q}_3$ associated with the directions x, y, z instead of the three scalar components a_x, a_y, a_z of a vector $\mathbf{a}$.

Likewise, just as the three scalars a_x, a_y, a_z associated with the co-ordinate directions define a vector $\mathbf{a}$, so the three *vectors* $\mathfrak{q}_1$, $\mathfrak{q}_2$, $\mathfrak{q}_3$ may be blended together to form a new kind of quantity (or magnitude) which we call a *tensor*. By the *components* of the tensor we mean the $3 \times 3 = 9$ components of the vectors $\mathfrak{q}_1$, $\mathfrak{q}_2$, $\mathfrak{q}_3$.

Since the direction cosines in (9), p. 6, or in (39), are the components of the unit vector $\mathbf{s}$, we can also write

$$\mathfrak{q} = \mathfrak{q}_1 s_x + \mathfrak{q}_2 s_y + \mathfrak{q}_3 s_z; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (39a)$$

for this we use the condensed notation

$$\mathfrak{q} = \mathrm{Q} \,.\, \mathbf{s},$$

and we say that the vector $\mathfrak{q}$ is produced by multiplication of the tensor Q by $\mathbf{s}$. We can also in the same way multiply Q by any other vector $\mathbf{r}$ of length $|\mathbf{r}| = r$; we thus obtain

$$\begin{aligned}\mathrm{Q} \,.\, \mathbf{r} &= \mathfrak{q}_1 r_x + \mathfrak{q}_2 r_y + \mathfrak{q}_3 r_z \\ &= r[\mathfrak{q}_1 \cos(\mathbf{r},\ x) + \mathfrak{q}_2 \cos(\mathbf{r},\ y) + \mathfrak{q}_3 \cos(\mathbf{r},\ z)]. \qquad (39b)\end{aligned}$$

Thus by means of the tensor Q every vector $\mathbf{r}$ is associated with a vector $\mathrm{Q} \,.\, \mathbf{r}$, the relation between the two vectors being linear and homogeneous.

In order to justify the introduction of tensors as magnitudes of a new kind, we may point to the fact e.g. that the "state of stress" at a given point of a solid body is represented by a tensor. Through a point P of the strained body we take an element of area dS, to which we assign a normal direction $\mathbf{s}$. If in imagination we remove the matter adjacent to dS on that side of it to which the vector $\mathbf{s}$ points, then in order to keep the remaining part of the body in position we must apply a certain force distributed over dS. Dividing this force by dS so as to make it refer to unit area, we obtain a certain *stress* $\mathbf{T}$ acting across the area dS, to which area we assign in the usual way a definite sense of description, or "currency". To every orientation of the element of area through P, and therefore also to every unit vector $\mathbf{s}$, there thus corresponds a stress vector $\mathbf{T}$. We now ask: how are these stress vectors, corresponding to the various directions $\mathbf{s}$, related to one another? Take a rectangular co-ordinate system with P as origin, and cut out of the corner of the first octant an infinitely small tetrahedron (fig. 16), the area of whose base is dS, and the outward normal $\mathbf{s}$ to which has the components $s_x = \cos\alpha$, $s_y = \cos\beta$, $s_z = \cos\gamma$. The areas of the other faces are then $dS\cos\alpha$, $dS\cos\beta$, $dS\cos\gamma$; the stresses acting on these are $-\mathbf{T}_1$, $-\mathbf{T}_2$, $-\mathbf{T}_3$, where $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ are the

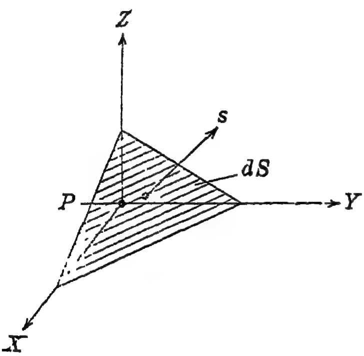

Fig. 16

respective stresses corresponding to the positive directions of the axes of x, y, z. (The outward normals of the faces are in the negative directions of the axes.) If the stress on the base is again denoted by $\mathbf{T}$, the forces acting on the tetrahedron are $\mathbf{T}dS$, $-\mathbf{T}_1 dS\cos\alpha$, $-\mathbf{T}_2 dS\cos\beta$, $-\mathbf{T}_3 dS\cos\gamma$. The equation of equilibrium is therefore

$$\mathbf{T}dS - \mathbf{T}_1 dS\cos\alpha - \mathbf{T}_2 dS\cos\beta - \mathbf{T}_3 dS\cos\gamma = 0;$$

and the relation required is

$$\mathbf{T} = \mathbf{T}_1\cos\alpha + \mathbf{T}_2\cos\beta + \mathbf{T}_3\cos\gamma. \quad . \quad . \quad . \quad (40)$$

Comparison with (39) shows that the state of stress at any one point of a solid body is a tensor; it is given if the stresses $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ corresponding to the directions of the axes are given for some co-ordinate system; the stress $\mathbf{T}$ across any area is a linear homogeneous function of these, and also of the direction cosines of the normal to the area. If T_x, T_y, T_z are the components of $\mathbf{T}$ parallel to the axes, then from (40)

$$\begin{aligned} T_x &= T_{11}\cos\alpha + T_{12}\cos\beta + T_{13}\cos\gamma, \\ T_y &= T_{21}\cos\alpha + T_{22}\cos\beta + T_{23}\cos\gamma, \\ T_z &= T_{31}\cos\alpha + T_{32}\cos\beta + T_{33}\cos\gamma; \end{aligned}$$

where e.g. T_{21} is the y-component of $\mathbf{T}_1$. In the special case of a stress tensor, these tensor components satisfy the condition of symmetry

$$T_{ik} = T_{ki},$$

the significance of which we shall see presently.

Another example of a tensor is obtained as follows.

We saw at equation (1a), p. 14, that the rate of increase of a scalar field ϕ in any direction is given by the corresponding component of a vector, viz. the gradient of ϕ:

$$\frac{\partial\phi}{\partial s} = \frac{\partial\phi}{\partial x}\cos(s, x) + \frac{\partial\phi}{\partial y}\cos(s, y) + \frac{\partial\phi}{\partial z}\cos(s, z).$$

If then a vector field is given, by every point of a region in space being associated with a vector $\mathbf{a}$, each of the components a_x, a_y, a_z, considered by itself, constitutes a scalar field. Applying (1a), p. 14, to these, we have

$$\begin{aligned} \frac{\partial a_x}{\partial s} &= \frac{\partial a_x}{\partial x}\cos(s, x) + \frac{\partial a_x}{\partial y}\cos(s, y) + \frac{\partial a_x}{\partial z}\cos(s, z), \\ \frac{\partial a_y}{\partial s} &= \frac{\partial a_y}{\partial x}\cos(s, x) + \frac{\partial a_y}{\partial y}\cos(s, y) + \frac{\partial a_y}{\partial z}\cos(s, z), \\ \frac{\partial a_z}{\partial s} &= \frac{\partial a_z}{\partial x}\cos(s, x) + \frac{\partial a_z}{\partial y}\cos(s, y) + \frac{\partial a_z}{\partial z}\cos(s, z). \end{aligned}$$

Multiplying these equations by the fundamental vectors $\mathfrak{i}$, $\mathfrak{j}$, $\mathbf{k}$ and adding, we obtain

$$\frac{\partial \mathbf{a}}{\partial \mathbf{s}} = \frac{\partial \mathbf{a}}{\partial x}\cos(\mathbf{s}, x) + \frac{\partial \mathbf{a}}{\partial y}\cos(\mathbf{s}, y) + \frac{\partial \mathbf{a}}{\partial z}\cos(\mathbf{s}, z). \quad . \quad (41)$$

While therefore the rate of increase of a scalar field defines a vector, the derivative of a vector field is given by a tensor.

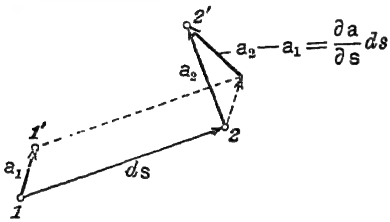

Fig. 17

We may make a physical application of this result. The most general distortion, or strain, of a body is defined when the displacement vector $\mathbf{a}$ is known for every point. Consider a small vector $d\mathbf{s}$ with components dx, dy, dz, connecting two particles of the body which is to be subjected to the strain. If $\mathbf{a}(x, y, z)$ is the displacement of the point (x, y, z), then the change in $d\mathbf{s}$ is the difference of the displacements of its ends, which is (fig. 17)

$$\frac{\partial \mathbf{a}}{\partial \mathbf{s}} ds = \frac{\partial \mathbf{a}}{\partial x} dx + \frac{\partial \mathbf{a}}{\partial y} dy + \frac{\partial \mathbf{a}}{\partial z} dz. \quad . \quad . \quad . \quad (41a)$$

According to (39a) this total differential is to be regarded as the product of the strain tensor (consisting of the partial derivatives of the displacement vector field) and the vector $d\mathbf{s}$.

If, following the notation used above for the stress tensor, we denote the components of a tensor by $q_{11}, q_{12}, q_{13}, q_{21}, \ldots$, so that q_{ik} is the ith component of that vector which corresponds to the kth co-ordinate axis, we can split up the vector equation (39) into the following equations in components:

$$\left.\begin{aligned} q_x &= q_{11}r_x + q_{12}r_y + q_{13}r_z, \\ q_y &= q_{21}r_x + q_{22}r_y + q_{23}r_z, \\ q_z &= q_{31}r_x + q_{32}r_y + q_{33}r_z. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (41b)$$

The tensor components, which are the coefficients of this linear system of equations, can be exhibited conveniently in the form

$$\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}.$$

This scheme is called a square *matrix*. The diagonal running downwards from left to right, which contains those elements of the matrix whose two indices are the same, is called the *principal diagonal*. If

elements which are images of each other in the principal diagonal are *equal* in each case, the matrix, as also the tensor whose components it contains, is said to be *symmetric*. Thus in a symmetric tensor we have

$$q_{12} = q_{21}; \quad q_{13} = q_{31}; \quad q_{23} = q_{32}.$$

If, on the other hand, the tensor components which are images of each other in the principal diagonal are *equal but of opposite sign*, and if the elements of that diagonal all vanish, the tensor is called *antisymmetric*, or *skew-symmetric*.

If we have two tensors A and B, which on multiplication by the vector $\mathbf{r}$ give the vectors

$$\mathbf{a} = \mathrm{A} \,.\, \mathbf{r} = \mathbf{a}_1 r_x + \mathbf{a}_2 r_y + \mathbf{a}_3 r_z$$

and

$$\mathbf{b} = \mathrm{B} \,.\, \mathbf{r} = \mathbf{b}_1 r_x + \mathbf{b}_2 r_y + \mathbf{b}_3 r_z,$$

then we have

$$\mathbf{a} + \mathbf{b} = (\mathbf{a}_1 + \mathbf{b}_1) r_x + (\mathbf{a}_2 + \mathbf{b}_2) r_y + (\mathbf{a}_3 + \mathbf{b}_3) r_z.$$

Hence the vectors $(\mathbf{a}_1 + \mathbf{b}_1)$, $(\mathbf{a}_2 + \mathbf{b}_2)$, $(\mathbf{a}_3 + \mathbf{b}_3)$, corresponding respectively to the three co-ordinate directions, define a tensor, whose product by $\mathbf{r}$ is $\mathbf{a} + \mathbf{b}$. We denote it by $\mathrm{A} + \mathrm{B}$, and call it the *sum* of the tensors A and B; its components are got by adding the corresponding components of A and B; its matrix is therefore

$$\begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{bmatrix}.$$

Any tensor whatever can be expressed as the sum of a symmetric and an antisymmetric tensor. For, by the addition rule, we have

$$\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} = \begin{bmatrix} q_{11} & \tfrac{1}{2}(q_{12} + q_{21}) & \tfrac{1}{2}(q_{13} + q_{31}) \\ \tfrac{1}{2}(q_{12} + q_{21}) & q_{22} & \tfrac{1}{2}(q_{23} + q_{32}) \\ \tfrac{1}{2}(q_{13} + q_{31}) & \tfrac{1}{2}(q_{23} + q_{32}) & q_{33} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & \tfrac{1}{2}(q_{12} - q_{21}) & \tfrac{1}{2}(q_{13} - q_{31}) \\ \tfrac{1}{2}(q_{21} - q_{12}) & 0 & \tfrac{1}{2}(q_{23} - q_{32}) \\ \tfrac{1}{2}(q_{31} - q_{13}) & \tfrac{1}{2}(q_{32} - q_{23}) & 0 \end{bmatrix}.$$

We have now to inquire what properties of a tensor are independent of the particular position the co-ordinate axes may chance to have. With this end in view, we write (41*b*) in the abbreviated form

$$q_i = \sum_{k=1}^{3} q_{ik} r_k \quad (i = 1, 2, 3), \quad . \quad . \quad . \quad (41c)$$

where we are denoting the x, y, z components of the vector $\mathbf{q}$ by q_1, q_2, q_3. If we now rotate the axes, so that q_i' and r_i' are the components of $\mathbf{q}$ and $\mathbf{r}$ with respect to the new co-ordinate system, there will be another relation

$$q_i' = \sum_s q'_{is} r_s' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41d)$$

between these quantities. The rotation of the axes is specified by the nine direction cosines a_{ik} of the new axes relative to the old. We have then

$$q_s' = \sum_s a_{is} q_s; \quad r_s' = \sum_k a_{sk} r_k. \quad . \quad . \quad . \quad . \quad (41e)$$

and the cosines a_{ik} are subject to the conditions of orthogonality

$$\sum_i a_{is} a_{ir} = \delta_{sr}, \text{ and } \sum_i a_{si} a_{ri} = \delta_{sr}, \quad . \quad . \quad . \quad (41f)$$

the symbol δ_{sr} standing for 0 when $s \neq r$, and 1 when $s = r$.

By inserting (41e) and (41c) in (41d), we obtain

$$\sum_{s,k} a_{is} q_{sk} r_k = \sum_{s,k} q'_{is} a_{sk} r_k.$$

This equation must hold for any vector $\mathbf{r}$. Hence we must have

$$\sum_s a_{is} q_{sk} = \sum_s q'_{is} a_{sk}.$$

On multiplying by a_{rk} and summing with respect to k, we find by (41f) the tensor transformation formula

$$q'_{ir} = \sum_{s,k} a_{is} a_{rk} q_{sk}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41g)$$

From (41g) we deduce at once: if for all indices $q_{sk} = q_{ks}$, then also $q'_{ik} = q'_{ki}$. On the other hand, if $q_{sk} = -q_{ks}$, then also $q'_{ik} = -q'_{ki}$. *The property of a tensor, of being symmetric or antisymmetric, is independent of the co-ordinate system.*

If in (41g) we put $i = r$ and sum for r, we find by (41f)

$$\sum_r q'_{rr} = \sum_s q_{ss}.$$

The sum of the elements in the principal diagonal of a tensor (the "spur") is therefore likewise an invariant.

By (41g), the δ-tensor δ_{ik} is not changed at all by rotation of the axes. Accordingly we also have

$$q'_{ir} - \lambda\delta_{ir} = \sum_{s,k} a_{is} a_{rk} (q_{sk} - \lambda\delta_{sk}),$$

where λ is an arbitrary number. Since the determinant of the cosines a_{ik} is equal to 1, the rule for the multiplication of determinants gives the important theorem: the value of the determinant

$$F(\lambda) = \begin{vmatrix} q_{11}-\lambda & q_{12} & q_{13} \\ q_{21} & q_{22}-\lambda & q_{23} \\ q_{31} & q_{32} & q_{33}-\lambda \end{vmatrix} \quad . \quad . \quad . \quad (41h)$$

is independent of the co-ordinate system. It follows that each of the coefficients of the polynomial $F(\lambda)$ is an invariant. *In particular, the roots* λ', λ'', λ''' *of the equation* $F(\lambda) = 0$, *called the "secular equation", are invariants, or have a meaning independent of axes.*

As an application we shall consider the rate at which the strain in a moving continuum changes with the time, the velocity $\mathbf{v}$ at every point being supposed given. We wish to find the change which takes place in the distance between two particles in time δt. Let two neighbouring particles be joined by the vector $\mathbf{s}$, whose components are x_1, x_2, x_3. Then the displacement of the initial point of $\mathbf{s}$ in time δt is $\mathbf{v}\,\delta t$, and that of the other end of $\mathbf{s}$ is $(\mathbf{v} + (\text{grad } \mathbf{v} \cdot \mathbf{s}))\delta t$. The components of the relative velocity of the particles are therefore given by

$$\dot{x}_i = \sum_k \frac{\partial v_i}{\partial x_k} x_k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41i)$$

The tensor $\dfrac{\partial v_i}{\partial x_k}$, which thus determines the rate of strain, may be separated into a symmetric and an antisymmetric part:

$$v_{ik} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right); \quad a_{ik} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i}\right). \quad . \quad (41k)$$

The symmetric part

$$\dot{x}_i = \sum_k v_{ik} x_k \qquad (v_{ik} = v_{ki})$$

has a characteristic property which may be brought out as follows. Let us try to find a vector whose direction is not changed; for this we must put $\dot{x}_i = \lambda x_i$. The system of equations

$$\sum_k v_{ik} x_k = \lambda x_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41l)$$

has a solution other than $x_i = 0$ if, and only if, λ is a root of the determinantal equation

$$F(\lambda) = | v_{ik} - \lambda\delta_{ik} | = 0.$$

To each of the roots λ', λ'', λ''' of this equation corresponds, by (41*l*), a direction x_i. If the roots are all different, then (41*l*) defines three

vectors which preserve their direction in the symmetric strain. In this case, if we take the equations

$$\sum_k v_{ik} x_k' = \lambda' x_i',$$

and

$$\sum_k v_{ik} x_k'' = \lambda'' x_i'',$$

multiply the first by x_i'' and the second by x_i', subtract, and sum for i, we obtain, in consequence of the symmetry of v_{ik},

$$0 = (\lambda' - \lambda'')(x_1'x_1'' + x_2'x_2'' + x_3'x_3'').$$

Thus the "proper vectors" x_i corresponding to two different values of λ are at right angles to each other. *The most general symmetric strain v_{ik} consists therefore of dilatations or compressions in three mutually perpendicular directions.* If the axes of co-ordinates are taken in these directions, then

$$\dot{x}_1 = \lambda' x_1; \quad \dot{x}_2 = \lambda'' x_2; \quad \dot{x}_3 = \lambda''' x_3.$$

The rate of change of the volume $V = x_1 x_2 x_3$ is given by

$$\begin{aligned} \frac{dV}{dt} &= \dot{x}_1 x_2 x_3 + x_1 \dot{x}_2 x_3 + x_1 x_2 \dot{x}_3 \\ &= V(\lambda' + \lambda'' + \lambda'''). \end{aligned}$$

But the sum of the three roots $\lambda' + \lambda'' + \lambda'''$ is the "spur" of the tensor (p. 48), and therefore invariant with respect to rotations of the axes (p. 48). Thus

$$\lambda' + \lambda'' + \lambda''' = v_{11} + v_{22} + v_{33} = \text{div}\,\mathbf{v}, \quad . \quad (41m)$$

and we see that the divergence is the "spur" of the tensor $\partial v_i/\partial x_k$.

Again, *the antisymmetric part a_{ik} (41k) of our tensor, for which*

$$\dot{x}_i = \sum_k a_{ik} x_k; \quad a_{ik} + a_{ki} = 0, \quad . \quad . \quad . \quad . \quad (41n)$$

leaves all lengths and angles unchanged. To prove this, consider the time rate of change of the scalar product of two arbitrary vectors x_i' and x_i'':

$$\begin{aligned} \frac{d}{dt}\sum_i x_i' x_i'' &= \sum_i (\dot{x}_i' x_i'' + x_i' \dot{x}_i'') \\ &= \sum_{i,k} a_{ik} x_i'' x_k' + \sum_{i,k} a_{ik} x_k'' x_i'. \end{aligned}$$

By interchanging the indices i and k in the last of these summations (a mere change of notation which does not alter the sum) we find

$$\begin{aligned} \frac{d}{dt}\sum_i x_i' x_i'' &= \sum_{i,k} x_i'' x_k' (a_{ik} + a_{ki}) \\ &= 0. \end{aligned}$$

from which the theorem stated above easily follows. The tensor a_{ik} must therefore represent a rotation. This may also be recognized at once, if we write

$$a_{12} = -u_3; \quad a_{13} = u_2; \quad a_{23} = -u_1. \quad . \quad . \quad . \quad (41o)$$

Equation (41*n*) then runs:

$$\dot{x}_1 = a_{12}x_2 + a_{13}x_3 = u_2x_3 - u_3x_2,$$
$$\dot{x}_2 = a_{21}x_1 + a_{23}x_3 = u_3x_1 - u_1x_3,$$
$$\dot{x}_3 = a_{31}x_1 + a_{32}x_2 = u_1x_2 - u_2x_1.$$

Comparison with (21*a*), p. 10, shows that (41*n*) does represent a rotation, and that the vector $\mathbf{u}$ defined in (41*o*) specifies the axis of rotation and the angular velocity. Further, by (41*k*) the vector $\mathbf{u}$ is connected with the velocity field $\mathbf{v}$ from which we started by the relation

$$\mathbf{u} = \tfrac{1}{2} \operatorname{curl} \mathbf{v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Both the angular velocity $\mathbf{u}$ *and curl* $\mathbf{v}$ *are properly speaking not vectors at all, but antisymmetric tensors.* The name "axial vectors" is sometimes used for magnitudes of this kind, to distinguish them from ordinary or "polar" vectors. Another important example of an axial vector is the vector product of two vectors a_i and b_i, which properly should be written as a skew-symmetric tensor

$$c_{ik} = -a_ib_k + a_kb_i.$$

The representation of a skew-symmetric tensor by a vector, as given in (41*o*), is only possible in three-dimensional space, and even then only in rectangular co-ordinates. In fact, as appears from (41*g*), the laws of transformation for vectors and antisymmetric tensors are not the same, unless the determinant of the cosines a_{ik} is equal to $+1$. Even the trifling modification of taking the determinant equal to -1 (signifying that the change of axes is from a right-handed to a left-handed system, or vice versa) is sufficient to bring out a difference in the nature of the two entities. Let us put, for example, $a_{ik} = -\delta_{ik}$ (i.e. take images in the origin). Then, according to (41*e*), the vector components become changed in sign while, by (41*g*), all tensor components (and accordingly axial vectors) remain unchanged.

In the preceding sections we may always recognize any axial vector that is introduced, from the fact that we require for its definition the idea of the right-handed screw. This idea, and the consequent restriction to right-handed co-ordinate systems, can be avoided altogether by always using the corresponding tensors instead of axial vectors.

To sum up, we may say that for three-dimensional space we have in the vector product a very simple and easily visualized practical

means of representing a skew-symmetric tensor. We must not forget, however, if we would avoid misunderstandings, that it amounts to no more than a practical rule, which is applicable only within a limited domain.

In applications to physics, the tensors with which we mainly have to do are *stress* tensors. In the theory of Elasticity the stress tensor arises from the symmetrical part of the strain tensor, by applying Hooke's Law. Since neither Hooke's Law nor the strain tensor has anything to do with a privileged sense of rotation, it follows that the stress tensor also is symmetric, as has already been mentioned (p. 45).

PART II

THE ELECTRIC FIELD

CHAPTER III

The Electrostatic Field in Free Space

1. Electric Intensity.

If a stick of sealing wax is rubbed with a piece of catskin, these bodies and the space round about them are thrown into a peculiar condition, as is revealed by the fact that light particles in the neighbourhood are set in motion. We say that the rubbed bodies are "electrified", and that the space surrounding them is an "electric field". The electrification is not irremovably fixed to the sealing wax and the catskin; it may be communicated to metals brought into contact with them. The process of rubbing is not the only means of producing the electrified state; a piece of metal which is in connexion with one of the poles of a battery also shows electrical actions, which continue even after the connecting wire is removed.

Let an electrified piece of metal be at rest in air. The electric field in its neighbourhood is investigated with the help of a *proof body*, which may be, for example, a small ball of elder pith covered with gold leaf, and electrified by contact with the rubbed sealing wax or the catskin. The proof body is acted upon by a force **F** in the electric field. Suppose this force **F** to be measured; both its magnitude and direction will be different at different points of the field; even at a fixed point in the field they will vary according to the way in which the pith ball is electrified. With regard to the latter type of variation, however, a very simple law governs the result; if the proof body was made to touch the sealing wax, then the direction and sense of the force **F**, which acts on it at a given point of the field, are perfectly definite, and only its magnitude depends on the details of the process; but if it was the catskin which was touched, then the direction of the force is reversed, its magnitude depending, as before, on the nature

of the preliminary process. This suggests that we should put, for the force which acts on the proof body in the electric field,

$$\mathbf{F} = e\,.\,\mathbf{E}, \quad \ldots \ldots \ldots \quad (1)$$

where the scalar e depends on the electrical state of the proof body, while the vector $\mathbf{E}$ is independent of that state, but varies in magnitude and in direction at the various points of the field. Experiment shows, in fact, that if two proof bodies which have been treated in different ways are brought in succession to the same point in the field, the forces upon them are in a definite ratio,

$$\mathbf{F}_1 : \mathbf{F}_2 = e_1 : e_2, \quad \ldots \ldots \ldots \quad (1a)$$

and that this ratio remains the same when the point varies. Experiment shows also that the magnitudes of the forces which act on one and the same proof body at different points P and P′ of the electric field are in a ratio independent of the previous treatment of the proof body, or

$$|\mathbf{F}| : |\mathbf{F}'| = |\mathbf{E}| : |\mathbf{E}'| \quad \ldots \ldots \quad (1b)$$

The statements (1*a*), (1*b*) are both included in (1). If e_1 is given for the first body, e_2 is defined for any other body by (1*a*); $\mathbf{E}$ can then be found for individual points of the field by means of *any* proof body.

The scalar factor e in the expression (1) *is called the "electric charge" of the proof body or the quantity of electricity upon it; the vectorial factor* **E** *is called the "electric intensity"*.* Charge and intensity are both specified unambiguously by equation (1), provided that the unit of charge has been defined. The opposite senses of the forces which act on the proof body, after contact with the sealing wax and the catskin respectively, can be taken into account by distinguishing two kinds of electricity, positive and negative. The positive sign has been given, quite arbitrarily, to the electricity on the little ball rubbed by the catskin, and consequently the negative sign to the rubbed sealing wax. The direction of the intensity $\mathbf{E}$ is accordingly defined as the direction of the force which acts on a proof body which has been brought into contact with the catskin.

The expression (1) for the force which acts on a charged body in the electric field, does not hold good without restriction. It ceases to hold exactly if the testing body is too close to the charged body, and that all the sooner the greater the charge on the small body. The expression becomes inexact in those cases also where the intensity varies too rapidly with change of position, and the more markedly so, the greater the dimensions of the proof body. Later on, we shall find out the causes of these divergences, and the expression for the force will be suitably completed in § 4, p. 91. Meantime, the proof

* This vector is also called the "electric force", and the "electric field strength".

body which we use must be sufficiently small and carry a sufficiently weak charge before we can determine the electric field by means of equation (1).

It is a characteristic feature of Maxwell's theory that it associates an intensity **E** with every point in space, and takes the vector field thus defined as the real subject of investigation. The physical significance of **E** consists to begin with *only* in the relation expressed in equation (1), which states: if a small charge e were brought up to the definite point of space in question, then the force $\mathbf{F} = e\mathbf{E}$ would act upon it there. Maxwell's theory then goes on to ascribe to this vector **E** a self-existent reality independent of the presence of a testing body. Although no observable force appears unless at least two charged bodies are present (for instance the charged piece of metal and the proof body), nevertheless we assert with Maxwell that the charged piece of metal by itself produces in the surrounding space a change of physical conditions which the field of the vector **E** is exactly fitted to describe. The primary cause of the action on a testing body is considered to be just this vector field *at the place where the testing body is situated.* As for the piece of metal, its part in the matter is merely to maintain this field. We speak accordingly of a theory of *field action,* as contrasted with the theory of *action at a distance* which was current before Faraday and Maxwell, and which starts from the mutual action between two charges.

2. Flux of Electric Force.

We consider in the first place the *electric field in free space, i.e. in a vacuum.* One of the most important results of quantitative electrical theory before Faraday was Coulomb's Law: two charged bodies, whose linear dimensions are small in comparison with their distance apart, act on each other with a force in the direction of the line joining them and inversely proportional to the square of the distance. Since either of the two bodies may be regarded at will as the proof body in the sense of the preceding section, we have

$$|\mathbf{F}| = \frac{e_1 \cdot e_2}{r^2} \cdot f,$$

where the factor f is independent of the nature and position of the bodies. Further, **F** is a repulsion if e_1 and e_2 have the same algebraic sign, but an attraction if their signs are different.

We can now introduce the *absolute electrostatic system* of measurement by laying down the definition: unit quantity of electricity is that quantity which acts upon an equal quantity at a distance of 1 centimetre with a force of 1 dyne. This implies that the factor f in the above equation has the value 1. It then follows from equation (1) that the unit of **E** is also determined.

We can now describe the result of Coulomb's experiments in the language of Faraday and Maxwell, as follows: a point charge of electricity e produces in its neighbourhood an electric field $\mathbf{E}$, which is given in magnitude and direction by the equation

$$\mathbf{E} = \frac{e}{r^2}\frac{\mathbf{r}}{|\mathbf{r}|}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1')$$

Here $\mathbf{r}$ is the vector drawn from the charge to the field point.

Comparison with our earlier results on point sources (p. 20) allows us to make the further statement: the electric field $\mathbf{E}$ due to a point charge e is identical with the irrotational velocity field which is produced in an incompressible fluid of density $1/4\pi$ by a point source of strength e.

In particular, if S be any closed surface enclosing the charge e, we have

$$\iint E_n dS = 4\pi e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We call $E_n dS$ the flux of force outwards across the surface element dS. Equation (2) therefore asserts that the flux of force outwards across a closed surface is equal to the point charge e within it, multiplied by 4π.

When several charges act simultaneously on a proof body, the forces they exert are superimposed according to the law of vector addition

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots .$$

In like manner, the intensities $\mathbf{E}_1$, $\mathbf{E}_2$ due to the charges are also superimposed by the vector law. This experimental fact allows us to give equation (2) the much more general interpretation: *the total flux of force outwards through a closed surface is equal to the total charge contained within the surface, multiplied by* 4π.

This theorem forms a suitable point of departure for Maxwell's theory. The electric charges no longer appear as centres of force, but as sources of flux of force.

We can now extend the results already obtained for surface and volume distributions of sources to the case of similar distributions of electricity.

A volume distribution of charge, the density of which is $\rho(x, y, z)$, produces divergence in the flux of force, given by

$$\operatorname{div} \mathbf{E} = 4\pi\rho.$$

A surface distribution of charge, of surface density ω, produces a discontinuity in the normal component * of $\mathbf{E}$, given by

$$-(E_{n_1} + E_{n_2}) = 4\pi\omega.$$

* Each normal component is taken in the direction *from* the field *to* the surface, as in fig. 7, p. 29.

3. **The Electrostatic Potential.**

The field due to a point charge, as defined in (1′), is irrotational. It can be expressed as the negative gradient of the scalar $\phi = e_1/r$:

$$\mathbf{E} = -\operatorname{grad}\phi; \quad \phi = \frac{e_1}{r};$$

$$E_x = -\frac{\partial\phi}{\partial x} = \frac{e_1}{r^2}\cdot\frac{x}{r} = \frac{e_1}{r^2}\cos(\mathbf{r}, x);$$

and similarly for E_y and E_z.

It follows that the general electrostatic field deduced by superposition from the volume sources $\rho\, dV$ and the surface sources $\omega\, dS$ is also irrotational:

$$\operatorname{curl}\mathbf{E} = 0. \quad \dots\dots\dots \quad (3)$$

It is therefore possible (p. 37) to represent the intensity of an electrostatic field as the negative gradient of a scalar, single-valued potential ϕ, or

$$\mathbf{E} = -\operatorname{grad}\phi, \quad \dots\dots\dots \quad (3a)$$

where ϕ is called "*the electrostatic potential*". The decrease of ϕ from a point (1) to a point (2) is equal to the line integral of $\mathbf{E}$, taken over an arbitrary path $\mathbf{s}$ from (1) to (2):

$$\phi_1 - \phi_2 = \int_1^2 \mathbf{E}\, d\mathbf{s}. \quad \dots\dots\dots \quad (3b)$$

The electrostatic field corresponds therefore in every respect to the field of an irrotational fluid motion, such as was discussed in an earlier section (p. 14). The strength e of the sources corresponds, by equation (2), to the quantity of electricity, which we have likewise denoted by e.

If the distribution of electricity is given, the electrostatic potential, and with it the irrotational field $\mathbf{E}$, are calculated by the methods explained in §§ 5–8 (p. 19). For a series of point charges, n in number, the potential (p. 20) is

$$\phi = \sum_{i=1}^{n} \frac{e_i}{r_i}; \quad \dots\dots\dots \quad (4)$$

for a volume distribution of charge

$$\phi = \int \frac{\rho\, dV}{r}, \quad \dots\dots\dots \quad (4a)$$

and for a surface distribution

$$\phi = \int \frac{\omega\, dS}{r}; \quad \dots\dots\dots \quad (4b)$$

the field of a double stratum would be found by the method indicated in § 9, p. 30.

4. The Distribution of Electricity on Conductors.

A problem in electrostatics is not as a rule a mere matter of being given a distribution of electricity, and being required to find the potential by means of (4*a*) and (4*b*). The distribution of electricity on metallic bodies is itself determined by certain conditions, to the definition of which we now proceed. We have already mentioned (p. 53) the property possessed by a metal wire, of communicating electricity to a body from the pole of a battery. A body which possesses this property is called a "*conductor of electricity*"; one which lacks it, an "*insulator*". The two classes cannot always be strictly separated.

The decision of the question whether a particular body is to be called a conductor or an insulator depends essentially on the period of time over which the observations extend. If we place the body in an electrostatic field, what happens in all cases is that a field is first set up in the interior of the body, so causing an electric current. This current tends to produce on the surface of the body a distribution of charge which in the interior of the body exactly compensates the external field. Supposing this state of affairs arrived at, we have again before us an electrostatic condition in which the field within the body is everywhere null. Now two extreme cases are possible. In the first case the time which elapses before this final condition is reached is small compared with the duration of the experiment (say 10^{-6} second), and consequently we shall find the field within the body to be null all the time. We then call the body a conductor. In the second case the time is very great (days or months). The current referred to then becomes so small that within the period usually occupied by an experiment it does not appreciably affect the results observed. In this case we speak of the body as an insulator. Pure electrostatics knows only idealized bodies, namely those in which the time referred to above is infinitely short ("metals"), and those in which it is infinitely long ("insulators"). Metals, in the sense of the word used in electrostatics, are therefore characterized by the property that in their interior the field **E** is null everywhere. In other words: *in the interior of a conductor the electrostatic potential ϕ is constant.*

The field produced in a region of space which contains charged metallic bodies, but is otherwise free of charge, can therefore be described as follows. At every point outside the metals we have

$$4\pi\rho = \operatorname{div} \mathbf{E} = 0.$$

In the *interior* of a metal there is obviously no charge, since there is no field there. There must, however, be a superficial distribution of sources on the outer *surface*, since a flux of force proceeds from it

outwards; this has the value E_n, where $\mathbf{n}$ is the normal directed into the outside space.

The surface density ω of the electricity, multiplied by 4π, is equal to the flux of force across unit area, i.e.

$$4\pi\omega = E_n = -\frac{\partial\phi}{\partial n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

There is still the possibility that, besides an ordinary surface distribution of sources of flux of force, there may be double sources on the bounding surface between air and metal. In fact, according to § 9 (p. 31), a homogeneous double stratum on this closed surface would make no change in the field, either inside or outside. For that reason it would be difficult to demonstrate its presence experimentally. We shall therefore in the meantime leave such double strata out of account.

If the irrotational field of the vector $\mathbf{E}$ is known, the distribution of electricity can be found from (5). If, on the other hand, the distribution of electricity on the surfaces of the conductors were known, then the field could be calculated from (4*b*) and (3*a*). Neither of these problems, however, is the one we actually have to face. The *fundamental problem of electrostatics* in fact is this: In space free from charge the electrostatic potential ϕ satisfies Laplace's equation:

$$\operatorname{div} \mathbf{E} = -\operatorname{div} \operatorname{grad} \phi = -\Delta\phi = 0. \quad . \quad . \quad . \quad (6)$$

On the surface S_i of any conductor, ϕ must take a constant value

$$\phi = \phi_i = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

In the interior of the conductor, ϕ must continue to have this value, since the gradient of the potential vanishes there. These matters being understood, the data are as follows. For each separate conductor we are given *either* its potential *or* its total charge,

$$e_i = \int \omega_i \, dS_i = -\frac{1}{4\pi}\int \frac{\partial\phi}{\partial n_i} dS_i; \quad . \quad . \quad . \quad (6b)$$

and we are required to find the corresponding solution of Laplace's equation. When this is known (an additive constant being possibly left arbitrary), the electric field is uniquely defined by the gradient of ϕ. This is the electrostatic field, and the corresponding distribution of electricity is the actual equilibrium distribution.

The fact that the field is uniquely defined by the above data may also be deduced from Green's theorem, which gives for the space bounded by the surfaces of the conductors

$$\iint \phi \frac{\partial\phi}{\partial n} dS = -\iiint |\operatorname{grad} \phi|^2 dV.$$

If there could be two possible solutions of the problem, ϕ_1 and ϕ_2, and if $\phi = \phi_1 - \phi_2$, we should have on every surface either $\phi = 0$ or $\int(\partial\phi/\partial n)dS = 0$, and accordingly $|\text{grad}\ \phi| = 0$ everywhere. This would mean, however, that ϕ_1 and ϕ_2 differ *at most* by an additive constant, and that only in the case when it is the charge e_i that is prescribed for *every* conductor. If the potential ϕ itself is prescribed for even *one* of the conductors, then the value of ϕ becomes absolutely definite everywhere.

5. Capacity of Spherical and Plate Condensers.

The problem of electrostatics has been solved in only a few cases. The simplest case is that of a charged metal sphere. Let e be its charge, and a its radius. We infer at once from the symmetry that the distribution of charge is uniform, so that

$$\omega = \frac{e}{4\pi a^2}$$

is the surface density of the electricity. We can satisfy equations (2) and (5), which express the relation between charge and flux of force, by assuming a flux of force equal to $4\pi e$ across every sphere concentric with the given conductor, the intensity accordingly being

$$E_r = \frac{e}{r^2}.$$

The potential of this irrotational field is

$$\phi = \frac{e}{r} + k;$$

on the sphere it has the constant value

$$\phi_a = \frac{e}{a} + k.$$

We must now, in order to obtain an electrostatic field which is physically possible, assign a terminus or sink for the flux of force issuing from the conductor. We shall assume that a second concentric hollow metal sphere of internal radius b encloses the first, and that its surface is charged with negative electricity. Since the charge $-e$ is distributed uniformly over this sphere, the surface density has the value

$$\omega = -\frac{e}{4\pi b^2};$$

and at $r = b$ the potential is

$$\phi_b = \frac{e}{b} + k.$$

This arrangement is called a *spherical condenser*; and the quotient of the positive charge e by the difference of potential $\phi_a - \phi_b$ of the positively and negatively charged conductors is called the "*capacity*" of the condenser. Since

$$\phi_a - \phi_b = e\left(\frac{1}{a} - \frac{1}{b}\right) = e\frac{b-a}{ab},$$

the capacity is
$$C = \frac{e}{\phi_a - \phi_b} = \frac{ab}{b-a}. \quad \quad (7)$$

Very large capacities can be produced by diminishing the distance $b - a$ between the two spheres.

When the capacity of a single sphere by itself is spoken of, it is implied that the other sphere which carries the opposite charge is situated at a very great distance; in this case the capacity of the sphere is equal to its radius a. In laboratory experiments the total quantity of electricity in a field is always nil. It is therefore necessary in each case to specify the position of the corresponding charge of opposite sign, i.e. the place where the flux of force from the sphere ends. In laboratory experiments the flux terminates on the walls of the room, or on the surface of some conductor in the room. If these are situated at distances which are great compared with the radius of the sphere, the capacity of the sphere is practically equal to its radius.

A *plate condenser* consists of two metallic plates the planes of which are parallel, and the distance between which is small compared with their lateral dimensions. If we neglect the scattering of the lines of force in the neighbourhood of the edge, we have between the plates a homogeneous field

$$|\mathbf{E}| = \frac{\phi_1 - \phi_2}{d},$$

and consequently a surface density ω of electricity given by

$$4\pi\omega = \frac{\phi_1 - \phi_2}{d}.$$

The capacity of a plate condenser in which the distance of the plates is d and the area of each is S is therefore

$$C = \frac{S}{4\pi d}.$$

This formula may be regarded as a special case of (7). If in fact the two radii a and b are very nearly equal, the spherical condenser may be looked upon as a plate condenser with plate distance $b - a = d$, and plate area $S = 4\pi ab$.

6. The Prolate Ellipsoid of Revolution.

We shall now consider a conducting prolate ellipsoid of revolution which is charged with electricity: what is its field, and what is the value of its capacity? The terminus of the flux of force proceeding from the conducting surface, if we are dealing merely with the capacity of the ellipsoid itself, is supposed to be at a very great distance; we may take it to be a sphere of large radius concentric with the ellipsoid. The mathematical problem may then be stated as follows (§ 4, p. 59).

Within the space between the two conductors the potential ϕ has to satisfy Laplace's equation

$$\Delta\phi = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

on the surfaces S_1, S_2 of the conductors it takes constant values,

$$\phi = \phi_1, \quad \phi = \phi_2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a)$$

the gradient of ϕ is perpendicular to these surfaces, and is proportional to the surface density ω, since, by (5), p. 59,

$$4\pi\omega = -\frac{\partial\phi}{\partial n}.$$

Here ω is still to a certain extent arbitrary, only the total charge e being assigned; where

$$e = \int \omega\, dS_1 = -\frac{1}{4\pi}\int \frac{\partial\phi}{\partial n_1}\, dS_1 = +\frac{1}{4\pi}\int \frac{\partial\phi}{\partial n_2}\, dS_2. \quad . \quad (8b)$$

As a rule the distribution of electricity is not so much what is wanted as simply the value of the capacity; this is defined when the potentials ϕ_1, ϕ_2 of the two conductors have been found; we then have

$$C = \frac{e}{\phi_1 - \phi_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8c)$$

Since no general method is known for solving the fundamental problem of electrostatics for conductors of arbitrary shape, we shall find the capacity of the prolate ellipsoid of revolution by a special method, only applicable to a conductor of this particular form. Going back to our hydrodynamical analogy, let us suppose the line joining the foci of the ellipsoid to be uniformly covered with sources. We shall prove that the equipotential surfaces of the corresponding irrotational field are confocal ellipsoids of revolution, and that this field also possesses the other properties required.

The total strength of the sources on the line, of length $2c$, we put equal to e. The potential of the line, viz.

$$\phi = \frac{e}{2c}\int_{-c}^{c}\frac{d\zeta}{r},$$

where r denotes the distance of the field point from any point on the line of sources, obviously is a solution of Laplace's equation (8). Taking the z-axis along the line of sources, and the origin at its mid point (fig. 1), we have

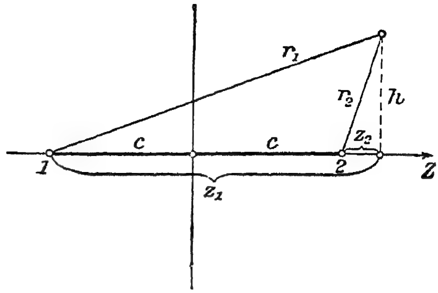

Fig. 1

$$r = \sqrt{\{(z-\zeta)^2 + x^2 + y^2\}},$$

and we obtain

$$\phi = -\frac{e}{2c}\Big|\log(z-\zeta+r)\Big|_{\zeta=-c}^{\zeta=+c}$$

$$= \frac{e}{2c}\log\frac{z+c+r_1}{z-c+r_2}, \quad \ldots \ldots \ldots \quad (9)$$

where r_1, r_2 are the distances of the field point from the ends of the line, at which $\zeta = -c$, $\zeta = +c$ respectively.

Writing for brevity

$$z + c = z_1, \quad z - c = z_2,$$

we must have for an equipotential surface, by (9),

$$\frac{z_1 + r_1}{z_2 + r_2} = k = \text{const.},$$

or

$$z_1 + r_1 = k(z_2 + r_2). \quad \ldots \ldots \quad (9a)$$

Dividing this by the obvious relation

$$r_1^2 - z_1^2 = r_2^2 - z_2^2 = h^2,$$

we get

$$\frac{1}{r_1 - z_1} = \frac{k}{r_2 - z_2},$$

or

$$k(r_1 - z_1) = r_2 - z_2,$$

and, on subtracting (9a) from this,

$$(k-1)(r_1 + r_2) = (k+1)(z_1 - z_2)$$
$$= (k+1)2c.$$

Hence
$$r_1 + r_2 = 2c\frac{k+1}{k-1},$$

i.e. the sum of the distances of a point on an equipotential surface from the two fixed points 1 and 2 is constant. The equipotential surfaces are therefore ellipsoids of revolution whose major axes, in terms of the parameter k, are
$$2a = r_1 + r_2 = 2c\frac{k+1}{k-1}.$$

At a very great distance $(a \to \infty)$ k becomes 1, so that $\log k = 0$, and therefore $\phi = 0$. Thus the potential vanishes on a sphere situated at a very great distance. If now we suppose one of the prolate ellipsoids of the confocal family to be a conductor, then the field in the space bounded on one side by this surface, and on the other by the very distant sphere, satisfies all the conditions of the electrostatic problem. This field is irrotational and contains no sources; and the total flux of force issuing from the ellipsoid is equal to the strength e of the charged line; lastly, the two conducting surfaces bounding the field are equipotential surfaces. Hence conditions (8), (8a), and (8b) are satisfied. Since by § 4 (p. 59) these conditions define the electrostatic field uniquely, ϕ is the potential of the required field.

From the equation for $2a$ we have
$$k = \frac{a+c}{a-c};$$

inserting this value in (9) we find
$$\phi = \frac{e}{2c}\log\frac{a+c}{a-c}$$
$$= \frac{e}{c}\log\frac{a+\sqrt{(a^2-b^2)}}{b}. \quad . \quad . \quad . \quad . \quad . \quad (9b)$$

Again, on the very distant sphere ($a = \infty$) we have
$$\phi_2 = 0;$$

hence the *capacity C of the prolate ellipsoid of revolution* is given by
$$\frac{1}{C} = \frac{\phi_1}{e} = \frac{1}{\sqrt{(a^2-b^2)}}\log\frac{a+\sqrt{(a^2-b^2)}}{b}. \quad . \quad . \quad (9c)$$

For a very long ellipsoid, i.e. for very small values of the fraction b/a, we obtain
$$\frac{1}{C} = \frac{1}{a}\log\frac{2a}{b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9d)$$

The capacity of such a rod-shaped conductor—which may be considered to be realized by a wire whose section is circular but which tapers towards the ends—diminishes with the thickness, for a given length. Further, the distribution of electricity in this limiting case is represented by the uniform distribution on the line between the foci, which we used above in order to determine the potential. Hence the electricity is distributed over the rod-shaped conductor in such a way that equal lengths of the wire carry equal charges.

7. A Point Charge in Front of a Conducting Plane.

We suppose the field to be bounded on one side by an infinite plane, which forms the surface of a conductor. At a point A, distant a from this plane, let a small body be placed, charged with e units of electricity. The dimensions of this body are to be so small, that its electric field, if the conducting plane were not there, would be derivable from the potential

$$\phi = \frac{e}{r}.$$

The question arises: how is the field affected by the conducting plane boundary? The above potential ϕ is obviously far from satisfying the condition of being constant on the conducting plane. We can, however, obtain a field for which that plane is an equipotential surface, by taking along with the point A another point B which is the image of A in the plane, and supposing a charge of contrary sign $-e$ to be placed at B. If r' is the distance of the field point from the image point, then

$$\phi = \frac{e}{r} - \frac{e}{r'} \quad . \quad . \quad . \quad (10)$$

represents the potential of the combined field in the half-space considered. This is zero on the boundary plane, since $r = r'$ there. The field is irrotational and, on the side on which A lies, solenoidal, i.e. free from sources, except at the point A itself; from A a flux of force issues, equal to $4\pi e$ (fig. 2).

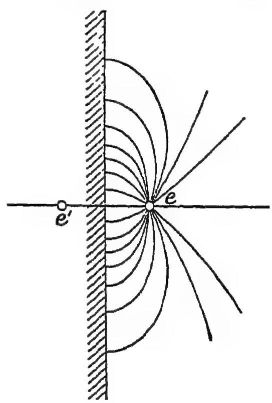

Fig. 2

On the plane the normal intensity, in the direction from the surface into the field, is

$$E_n = -\frac{\partial \phi}{\partial n} = -e\frac{\partial}{\partial n}\frac{1}{r} + e\frac{\partial}{\partial n}\frac{1}{r'}$$

$$= -\frac{2ae}{r^3};$$

and the surface density ω is therefore, by (5), p. 59,

$$\omega = \frac{1}{4\pi} E_n = -\frac{e}{2\pi}\frac{a}{r^3}. \quad . \quad . \quad . \quad . \quad . \quad (10a)$$

Hence electricity distributes itself on the plane surface of the conductor in such a way that the surface density is inversely proportional to the cube of the distance from the point charge. For the total charge on the plane we have

$$\int \omega dS = -\frac{e}{2\pi}\int \frac{a\,dS}{r^3},$$

or, by introducing polar co-ordinates ρ, θ,

$$\int \omega dS = -\frac{e}{2\pi}\int_0^{2\pi} d\theta \int_0^\infty \frac{a\rho\,d\rho}{(a^2+\rho^2)^{\frac{3}{2}}}$$

$$= +e\left|\frac{a}{(a^2+\rho^2)^{\frac{1}{2}}}\right|_0^\infty = -e.$$

Accordingly the whole of the flux of force which begins at A ends on the plane surface of the conductor. At the point where the charge e is placed, the intensity due to the charge on the plane is identical with that which would be produced by the image $e' = -e$. Hence the force acting on e is the "*image force*"

$$\frac{e^2}{(2a)^2}.$$

This phenomenon, in which an electrically charged body calls forth a charge of opposite sign on the surface of a neighbouring conductor, originally uncharged, is called "*electrostatic induction*". The phenomenon may be regarded as a consequence of the fact that the field cannot penetrate into the interior of the conductor. If the conductor is of finite dimensions and has no conducting connexion with other bodies, then, seeing that its total charge continues to be nil, the flux of force which reaches it on the side facing the exciting point must leave the conductor again on the other side. In the case discussed above of a conductor extending to infinity and cutting out the field on one side, we must regard the charge $+e$, which was produced at the same time as the induced charge $-e$ when the exciting point was brought up, as having been removed to infinity.

When a point charge, say a small charged body used for the purpose of exploring a field, is brought into the neighbourhood of a charged conductor, its field, as influenced by the presence of the conductor, is superimposed upon the original field of the conductor. Hence the actual force upon the testing body will not correspond

to the original distribution on the conductor, but to the distribution as changed by the presence of the testing body itself. It follows that the force will not give an exact measure of the original field, and that it will be less and less correct the greater the charge on the testing body and the nearer it is brought to the conductor. In the immediate neighbourhood of the conducting surface the method of finding the vector **E** given in § 1, p. 54, is only correct when the charge on the small body can be made infinitely small. Strictly speaking, the vector **E** is not defined by any special value of the ratio of the force on the small body to its charge, but by the *limit* to which this ratio approximates when the charge e is indefinitely diminished.

8. Point Charge and Conducting Sphere.

Before proceeding to investigate the *charge induced on a conducting sphere,* we shall first consider the following problem: given two charges e_1 and $-e_2$ at a given distance apart, it is required to find the surface on which the potential

$$\phi = \frac{e_1}{r_1} - \frac{e_2}{r_2}$$

has the value zero. Let e_2 be the absolutely smaller of the two charges.

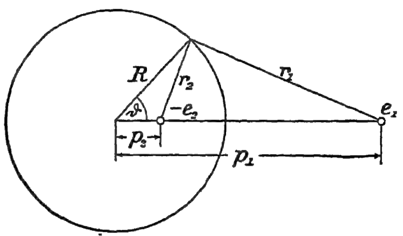

Fig. 3

We take as origin of polar co-ordinates R, θ a point on the prolongation of the line $e_1 \to e_2$, and denote its distances from the two charges by p_1 and p_2. We have then (fig. 3)

$$r_1^2 = R^2 + p_1^2 - 2Rp_1 \cos\theta,$$
$$r_2^2 = R^2 + p_2^2 - 2Rp_2 \cos\theta.$$

The potential is therefore zero if

$$\frac{e_1^2}{e_2^2} = \frac{r_1^2}{r_2^2}$$
$$= \frac{p_1}{p_2} \frac{R^2/p_1 + p_1 - 2R\cos\theta}{R^2/p_2 + p_2 - 2R\cos\theta}.$$

We see that this relation is satisfied for all values of θ, if

$$\text{(i)}\ R^2 = p_1 p_2,$$

and

$$\text{(ii)}\ \frac{p_1}{p_2} = \frac{e_1^2}{e_2^2}.$$

The potential is therefore zero on a sphere, whose centre divides the

line joining the two point charges externally in the ratio of the squares of the charges; and whose radius is such that the points where the charges are placed are "inverse" to each other with respect to the sphere.

Point Charge and Metallic Sphere.—A charge e is situated at a distance p from the centre of a conducting sphere of radius R. We shall consider two problems. In the first, the sphere is kept at potential zero (by being connected to earth). A glance at fig. 3 allows us to write down the solution at once. Thus, suppose the sphere removed, and in its stead a point charge

$$-e' = -e\sqrt{\frac{p'}{p}} = -e\frac{R}{p}$$

placed at a distance

$$p' = \frac{R^2}{p}$$

from its centre. This, along with the given point charge, produces a field whose potential over the original spherical surface has the zero value required, and outside this surface has only the single source e. The potential outside the earthed sphere is therefore defined by *

$$\phi = \frac{e}{r} - \frac{e'}{r_1}.$$

In the second problem, the sphere is insulated, and previous to the point charge being brought up was uncharged; hence of course it remains uncharged throughout. In order to describe its field we must therefore imagine a further charge $+e'$ placed within it, in such a position that the constancy of the potential on the surface is not disturbed thereby, i.e. we suppose a charge $+e'$ placed at the centre of the sphere (fig. 4). The potential of the combination,† point charge e and insulated uncharged sphere, is then

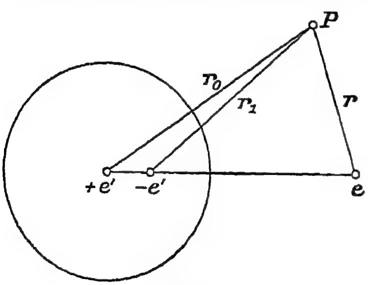

Fig. 4

$$\phi = \frac{e}{r} - \frac{e'}{r_1} + \frac{e'}{r_0},$$

* [The total charge induced on the metallic sphere is $-e' = -eR/p$, since the total flux of force ending on the earthed sphere is the same as the flux which would end at the image if the sphere were not present.]

† [The total charge on the sphere is (see preceding footnote) $-e' + e' = 0$, as it should be.]

where r_0 denotes the distance of the field point from the centre. The potential of the spherical surface itself is now e'/R, or e/p, i.e. it is equal to the potential at the centre due to e alone.

It is interesting to see what happens if we send the point charge to infinity, while increasing it at the same time in such a way that the field it produces, viz.

$$|\mathbf{E}_0| = \frac{e}{p^2},$$

retains a finite value. In this process the image point $-e'$ goes, of course, to the centre of the sphere, while

$$e'p' = e\frac{R^3}{p^2}$$

retains the finite value $|\mathbf{E}_0| \,.\, R^3$. We therefore obtain a double source or, as we say, an *electric dipole* at the centre of the sphere, which is defined vectorially by the relation

$$\mathbf{m} = \mathbf{E}_0 \,.\, R^3.$$

The field $\mathbf{E}_0$ of the infinitely distant and infinitely great point charge is, of course, in the neighbourhood of the sphere, homogeneous: *an insulated conducting sphere in a homogeneous electric field becomes polarized in such a way that its surface charge acts in the exterior space like a dipole of moment $\mathbf{E}_0 R^3$ supposed to be placed at the centre of the sphere.*

CHAPTER IV

Dielectrics

1. The Plate Condenser and the Dielectric.

Up to this point we have confined ourselves to the electric field in a vacuum (free space). When, as we have done occasionally, we have spoken of the field in air, we have been guilty of a slight want of exactness which, however, as we shall see immediately, is usually unimportant. We have therefore to state explicitly now that the formulæ of the preceding chapter are to be considered as referring to a vacuum and to metals placed in a vacuum.

The fundamental discovery was made by Faraday that the capacity of a condenser is altered when the space between its conducting faces is occupied by an insulator, such as glass or sulphur or petroleum. With any known material so used, the capacity in fact is increased. The factor K, by which C is thus multiplied, was found to be a constant which is characteristic of the interposed substance. It is called the "*dielectric constant*" of the material in question.* Hence, by § 5, p. 60, we now have as values of the capacities:

$$\text{spherical condenser,}\quad C = K\frac{ab}{b-a};$$

$$\text{plate condenser,}\quad C = K\frac{S}{4\pi d}.$$

The following are examples of numerical values of K:

Air	1·0006	Porcelain	6
Sulphur dioxide	1·01	Alcohol	26
Petroleum	2·0	Water	81
Glass	5 to 7		

In a vacuum K has by definition the value 1. Instead of the word vacuum we also occasionally use the word "æther", not connecting it in any way with the idea of a hypothetical substance, but merely using the word when we are speaking of space as the carrier of an electromagnetic field.

*The name "specific inductive capacity" is also frequently used.

We shall begin by trying to form a clear picture of the nature of Faraday's discovery in the light of the processes in a plate condenser.

Let the two opposing plane faces of the condenser, at distance d from each other, be maintained in all cases (say by means of a battery) at the constant potential difference $\phi_1 - \phi_2$. Then in a vacuum the field between the plates (directed downwards in fig. 1) has everywhere the constant value $|\mathbf{E}_0| = (\phi_1 - \phi_2)/d$, so that on each plate there is a charge of surface density $\pm\omega_0$, where

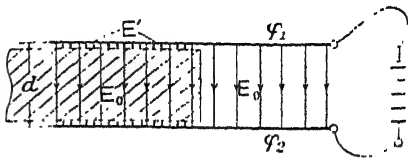

Fig. 1

$$4\pi\omega_0 = |\mathbf{E}_0| = \frac{\phi_1 - \phi_2}{d}.$$

If we now insert in the condenser a plate of insulating material, of thickness d and dielectric constant K, we obtain in that part of the condenser which is occupied by the material a different surface density of charge, viz.

$$4\pi\omega = K\,|\mathbf{E}_0| = K\frac{\phi_1 - \phi_2}{d}.$$

While the plate is being inserted in the condenser, the battery must accordingly supply a quantity of electricity

$$\omega - \omega_0 = \frac{K-1}{4\pi}\,|\mathbf{E}_0|$$

for every square centimetre which is covered by the insulator; that it actually does so may easily be verified with the help of an amperemeter. For the success of this experiment, it is quite unnecessary that the insulator (say a glass plate) should come into contact with the faces of the condenser. The experiment comes off exactly as before if we leave a narrow opening between metal and insulator, provided only the width of the opening is small compared with the distance d between the plates. Now it is proved by experiment that such an opening does not alter the capacity or, consequently, the charge of the condenser; hence in the opening the intensity must have the value

$$4\pi\omega = |\mathbf{E}'| = K\,|\mathbf{E}_0|,$$

for the metal surface has now a vacuum adjacent to it. In the interior of the insulator, on the other hand, the intensity must still be $\mathbf{E}_0$, since the line integral $\int E_s ds$ taken from the one plate of the condenser to the other must in both cases have the value $\phi_1 - \phi_2$. It follows that when we pass from the opening into the insulator, the intensity

$\mathbf{E}$ jumps from $K\mathbf{E}_0$ to $\mathbf{E}_0$. But a sudden change in the normal component of the field strength is always equivalent to the presence of a surface charge. The effect of the insulator on the electrostatic field is therefore the same as if its surface carried a charge of surface density ω', where

$$4\pi\omega' = (K-1)(\mathbf{E}_0\mathbf{n}),$$

$\mathbf{E}_0$ being the intensity in the insulator, and $\mathbf{n}$ the normal to the insulator, drawn outwards.

2. Dielectric Polarization.

The property possessed by an insulator, which as a whole is uncharged, of affecting the field in this way, is called "*polarizability*". The insulator is "polarized" by the electric field $\mathbf{E}_0$. To understand this property, we must make the assumption that every material body contains positive and negative electricity (charges), and in fact an equal quantity of each kind if the body is electrically neutral. But while in a conductor one at least of the two kinds is freely movable (electrons in a metal, ions in an electrolyte), in an insulator the two kinds are bound quasi-elastically, and that in such a way that under the influence of an electric field the charges can move slightly (the positive in the direction of the field or the negative in the opposite direction or both at once); the motion, however, coming to a standstill when the displacement has reached a certain amount proportional to the strength of the field. When the field is switched off, the displaced electricity returns to its original position. This relative displacement of charges is called *polarization*; it is measured by a vector $\mathbf{P}$, defined as follows. Starting from unpolarized material, we take within it an arbitrary direction $\mathbf{s}$, and an element of area dS perpendicular to $\mathbf{s}$. If the material now becomes polarized, the quantity of electricity which on the whole passes through dS in the direction $\mathbf{s}$ in consequence of the polarization is equal to the component of $\mathbf{P}$ in the direction $\mathbf{s}$ multiplied by the area dS.

A body is said to be homogeneously polarized if the vector $\mathbf{P}$ is the same throughout the body. It is clear that residual or surplus charges due to polarization can only arise in a given part V of the volume of the body, provided the surface integral

$$e' = -\int_V P_n dS,$$

taken over the surface enclosing V, has a value different from zero, $\mathbf{n}$ being the outward normal. In fact $(-P_n dS)$ is the charge which in this case is displaced from the outside through dS into V. Hence, by Gauss's theorem,

$$\rho' dV = -\operatorname{div} \mathbf{P} dV$$

is the charge arising in the element of volume dV in consequence of the polarization.

It follows that no internal charges can occur when the polarization is homogeneous. On the other hand, electricity in the form of surface charges will appear at the bounding surface. To see this, consider a flat cylinder with base dS, one plane face of which lies wholly in the outside vacuum, but the other wholly within the polarized body. The charge in this cylinder due to polarization is

$$\omega' dS = P_n dS.$$

The surface of a polarized body therefore carries a surface charge the density ω' of which is equal to P_n.

We can arrive at the same formulæ for ρ' and ω' in another way as follows. Suppose the dielectric divided up into small cylindrical elements of volume $dV = dS \,.\, h$, the bases dS of which are perpendicular to $\mathbf{P}$. The polarization of dV gives rise to an electric dipole with the moment

$$\mathbf{m} = \mathbf{P} dV,$$

for charges $\pm |\mathbf{P}| \, dS$ occur on the ends of the little cylinder, and these being at distances h from each other produce a moment $|\mathbf{P}| \,.\, dS \,.\, h$. The potential ϕ' called forth by this dipole at a field point at distance r is, by (20′), p. 23, given by

$$\left(\mathbf{m} \,.\, \mathrm{grad}_s \frac{1}{r}\right),$$

and the potential due to the polarized body is therefore on the whole

$$\phi' = \iiint \left(\mathbf{P} \,.\, \mathrm{grad}_s \frac{1}{r}\right) dV.$$

But $\mathrm{div}\left(\frac{\mathbf{P}}{r}\right) = \frac{1}{r} \mathrm{div}\, \mathbf{P} + \left(\mathbf{P} \,.\, \mathrm{grad} \frac{1}{r}\right)$, so that by Green's theorem we obtain

$$\phi' = \iint \frac{P_n dS}{r} - \iiint \frac{\mathrm{div}\, \mathbf{P}}{r} dV.$$

This equation, however, simply states that the insulator carries a surface charge of density $\omega' = P_n$, and a space charge of volume density $\rho' = -\mathrm{div}\, \mathbf{P}$.

We have therefore two perfectly equivalent definitions of $\mathbf{P}$: (1) *the electric moment per unit volume*, (2) *the quantity of electricity passing through a unit of area which is perpendicular to* $\mathbf{P}$.

In the case of the plate condenser discussed above (fig. 1, p. 71) we have clearly to do with a homogeneous polarization of the inserted plate, in which the vector $\mathbf{P}$ is directed downwards, and

is given in amount by the free charge on the surface of the plate, or

$$\omega' = \mathbf{P} = \frac{K-1}{4\pi}\mathbf{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Thus the numerical relation between the vectors $\mathbf{P}$ and $\mathbf{E}$ is defined by means of the dielectric constant K. The factor of proportionality which appears in (1), viz.

$$\chi = \frac{K-1}{4\pi},$$

is called the "dielectric susceptibility" of the material in question.

3. Maxwell's Displacement Vector D.

The space charges ρ' and surface charges ω' due to polarization produce on their part a corresponding divergence of the field intensity (p. 56)

$$\left.\begin{aligned} \operatorname{div} \mathbf{E} &= 4\pi\rho' = -4\pi \operatorname{div} \mathbf{P} \\ \text{and} \quad -(E_{n_1} + E_{n_2}) &= 4\pi\omega' = 4\pi(P_{n_1} + P_{n_2}). \end{aligned}\right\} \quad . \quad . \quad (1a)$$

In words: the vector $\mathbf{E} + 4\pi\mathbf{P}$ in an insulator is everywhere solenoidal. Its normal component is subject to no discontinuity at the boundary between two insulators.

For the vector thus brought to notice we introduce (after Maxwell) the special name "*electric displacement*" and the symbol $\mathbf{D}$, so that

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This vector is therefore characterized by the following properties:

1. In the interior of an uncharged insulator $\operatorname{div} \mathbf{D} = 0$ everywhere.
2. At the boundary between two insulators the normal component of $\mathbf{D}$ is everywhere continuous.

The statement 2 is of course really only a consequence of 1, since the behaviour at the bounding surface can always be deduced from the condition $\operatorname{div} \mathbf{D} = 0$ by assuming a continuous transition from the one insulator to the other, and passing in a suitable way to the limit.

Closely connected with the introduction of the vector $\mathbf{D}$ we have the ideas of "true" and "free" charges, which play a great part in older expositions of electrodynamics. *Free charges are defined as sources of* $\mathbf{E}$, *true charges on the other hand as sources of* $\mathbf{D}$. If we put an insulated metal sphere, charged with a quantity e of electricity, in a homogeneous dielectric, its charge becomes partly compensated by the surface charge $\omega' = P_n$ of the adjacent part of the dielectric, so that only the free charge e/K is now available as a source of $\mathbf{E}$. The true charge on the contrary remains unchanged in this process.

For, according to its definition, it is found from the free charge by subtracting the space charges ρ' and the surface charges ω' which are due to polarization. *When in future we speak simply of charges, we shall always mean true charges, that is, sources of* **D**. From these definitions we have the following results:

3. In the presence of true charges (which must therefore be charges on metallic bodies, or charges otherwise introduced into the insulator), the surface integral

$$\frac{1}{4\pi}\int\int D_n dS$$

gives the total charges contained within S. In particular, therefore, the surface density ω of the charge on a metal contiguous to an insulator is given by the normal component of **D**, or

$$4\pi\omega = D_n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For a volume density of charge,

$$4\pi\rho = \operatorname{div} \mathbf{D}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

4. In an isotropic insulator, by (1) and (2),

$$\mathbf{D} = K\mathbf{E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where K may be an arbitrary function of position, and the electric intensity **E** is everywhere irrotational:

$$\mathbf{E} = -\operatorname{grad} \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

in a region where K is continuous, and the tangential components of **E** are continuous on the two sides of surfaces where K is discontinuous.

In many accounts of the subject, the vector **D** is immediately defined by (4). It must therefore be expressly remarked that this definition is much more special than that given by (2). It only holds when the polarization **P** is proportional to the intensity **E**. In crystalline media, however, this is not the case. In these the direction of **P** is in general different from that of **E**, and in place of the scalar dielectric constant K, we have a tensor K_{ik},

$$D_i = \sum_k K_{ik} E_k \qquad (i, k = x, y, z).$$

Thus even in this case there is still a linear relation between **D** and **E**. We shall see later that in the perfectly analogous case of magnetic induction and magnetic intensity in ferromagnetism, no such relation

exists. On the other hand, equation (2) can be transferred to the magnetic case just as it stands.

The boundary conditions at the surface where two insulators meet (continuity of the normal component of **D** and of the tangential component of **E**) give rise to a special law of refraction for the lines of force. Let α_1 and α_2 be the angles between the line of force and the normal on the two sides of the surface of discontinuity (fig. 2); the conditions are then

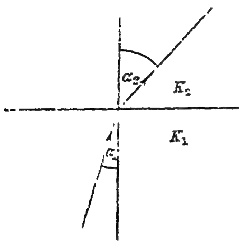

Fig. 2

$$|\mathbf{D}_1|\cos\alpha_1 = |\mathbf{D}_2|\cos\alpha_2,$$

and
$$|\mathbf{E}_1|\sin\alpha_1 = |\mathbf{E}_2|\sin\alpha_2,$$

which, combined with (4), give at once

$$\tan\alpha_1 : \tan\alpha_2 = K_1 : K_2.$$

It follows that the lines of force, when they pass from one insulator into another for which K is greater, are deflected farther from the normal.

4. Spherical Condenser. Semi-infinite Dielectric.

1. Let the space (fig. 3) between the two spherical surfaces of a condenser be filled, in concentric layers, with two different substances whose dielectric constants are K_1 and K_2. Let a and c be the radii of the inner and the outer spherical surfaces, b the radius of the sphere forming the boundary between the two media, $+e$ and $-e$ the respective charges on the inner and the outer sphere. Then in the space between a and c we have

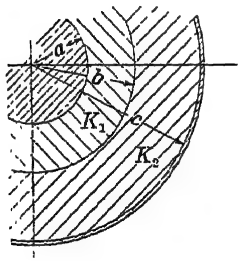

Fig. 3

$$D_r = \frac{e}{r^2};$$

but, from a to b,
$$E_r = \frac{1}{K_1}\frac{e}{r^2};$$

from b to c,
$$E_r = \frac{1}{K_2}\frac{e}{r^2}.$$

Accordingly, the potential difference between a and c is

$$\phi_a - \phi_c = \int_a^c E_r dr$$
$$= \frac{e}{K_1}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{e}{K_2}\left(\frac{1}{b} - \frac{1}{c}\right).$$

For the capacity C we have therefore

$$\frac{1}{C}=\frac{1}{K_1 a}-\frac{1}{K_2 c}+\frac{1}{b}\left(\frac{1}{K_2}-\frac{1}{K_1}\right).$$

The case of a sphere of radius a embedded in a dielectric envelope of radius b is found by putting $K_2=1$, $c=\infty$; hence, in this case,

$$\frac{1}{C}=\frac{1}{K_1 a}\left\{1+\frac{a}{b}(K_1-1)\right\}.$$

If b is large compared with $a(K_1-1)$, the envelope practically acts as if the sphere were placed in infinite space of dielectric constant K_1.

2. *A point charge in front of a semi-infinite dielectric.*

Suppose we have a charge at the point A in air, at a distance a from the plane surface of a dielectric; what is the effect produced by the presence of the dielectric? This problem corresponds to the one solved in § 7, p. 65, for the conducting plane. In that case, however, we had only to take the field in air into account, since on the far side of the conducting plane no field existed at all. But now we shall have to consider the field within the insulator. We shall assume the insulator to occupy all space beyond an infinite bounding plane. Let its dielectric constant be K_2, and that of the air K_1.

We shall try to solve this problem, as we did the other, by the method of electrical images. As before, we take the point B within the dielectric, where B is the optical image of A in the bounding plane; and we denote by r, r' the distances of the field point from A and B.

We put for the potential in the air space

$$\phi_1=\frac{e}{K_1 r}-\frac{e'}{K_1 r'}.$$

The field in air is therefore to correspond to the true charge e at A, and the true charge $(-e')$ imagined at B. This assumption satisfies the fundamental condition that sources of electrical displacement must occur at A only; for the image point B is outside the air space. As for the field within the dielectric, we shall try to represent it by putting for the potential in the dielectric

$$\phi_2=\frac{e''}{K_2 r}.$$

The field in the insulator is thus to be the same as if the insulator occupied all space and a true charge e'' were placed at A. This is in agreement with the condition that within the part of space actually occupied by the dielectric there are no sources or sinks of electrical displacement.

To prove that the field is actually represented correctly by these

assumptions, we shall show that the boundary conditions at the plane separating the two media can be satisfied by a suitable choice of the quantities e', e'', which so far have not been defined. Considering first the normal components of **D**, we have

$$D_{n_1} = +e\frac{a}{r^3} + e'\frac{a}{r'^3}, \qquad D_{n_2} = -e''\frac{a}{r^3},$$

where the normals are taken in the directions from within the body concerned to the surface. One boundary condition is that the normal component of **D** is continuous; since $r = r'$ on the plane, this condition requires

$$e + e' - e'' = 0.$$

Again, the tangential components of **E** on the two sides of the separating plane are to have equal values; this will certainly be the case if $\phi_1 = \phi_2$ all along the plane, for **E** is simply the negative gradient of ϕ. The condition $\phi_1 = \phi_2$ is not only sufficient, it is also necessary if double strata of free electricity on the surface of the insulator are excluded. We therefore require secondly

$$\frac{e - e'}{K_1} = \frac{e''}{K_2}.$$

From these two linear equations connecting e, e', e'' we find

$$\left.\begin{aligned} \frac{e-e'}{e+e'} = \frac{K_1}{K_2}, \quad e' = e\frac{K_2 - K_1}{K_2 + K_1}, \\ e'' = e + e' = e\frac{2K_2}{K_2 + K_1}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

The suppositious "true" charges, $(-e')$ at B and $(+e'')$ at A, are therefore determined. The lines of force within the dielectric run so that they appear to diverge radially from A, while in the air space the field is the resultant of the two fields arising from the source A and the sink B.

If the dielectric is replaced by a conductor, the image point B must, by § 7, p. 65, be given the charge $(-e)$ in order to define the potential in the air space. The disturbing action of the dielectric on the field intensity in air, and the disturbing action of the conductor, are to one another in the ratio

$$e' : e = K_2 - K_1 : K_2 + K_1.$$

The influence of the dielectric is therefore always slighter than that of the conductor. In the limiting case, however, when the dielectric constant K_2 of the insulator is very great compared with that of air, e becomes equal to e'; that is to say, *the conductor affects the field*

in air in the same way as an insulator whose dielectric constant is infinitely great.

As for the field strength within the dielectric, this corresponds to a free charge e''/K_2 placed at A in an unlimited dielectric; if the dielectric were removed, the intensity in the space it occupied would be defined by the free charge e/K_1 which is actually placed at A in air. The effect of the field is accordingly measured by the ratio

$$\frac{e''}{K_2} : \frac{e}{K_1} = 2K_1 : K_2 + K_1.$$

In the limiting case of an infinite dielectric constant, K_2, the electric intensity in the interior of the insulator is nil, exactly as in the interior of a conductor.

5. Dielectric Sphere in a Homogeneous Field.

We consider a sphere of radius a, and dielectric constant K_1, embedded in another dielectric occupying the whole of the rest of space; in the latter dielectric we suppose a homogeneous field $E_x = E_0$, in the positive direction of the x-axis, to have been in existence before the introduction of the sphere. How is this field modified by the introduction of the sphere? In order to answer this question we define a potential ϕ having the following properties:

1. At a great distance from the sphere ($r \to \infty$) ϕ must become $(-)E_0x$.
2. At the surface of the sphere, the normal component of the gradient of ϕ undergoes a sudden change, such that $K(\partial\phi/\partial n)$ has the same value on the two sides.
3. ϕ itself, and therefore the tangential component of grad ϕ, are continuous on the two sides of the sphere.
4. ϕ everywhere satisfies Laplace's equation $\Delta\phi = 0$.

We denote by ϕ_1 and ϕ_2 the values of the potential inside and outside the sphere respectively, and try putting

$$\phi_1 = -Fx$$
$$\phi_2 = -E_0x + E_0\frac{k}{r^2}\cdot\frac{x}{r}.$$

The interpretation of these equations is: In the interior (ϕ_1) of the sphere there exists a homogeneous field **F**, so that the sphere is homogeneously polarized in the x-direction. In the outer space (ϕ_2), on the other hand, the sphere acts as if a dipole of moment E_0k were situated at its centre.

The first and fourth of the conditions laid down are obviously satisfied by the forms assumed for ϕ. We have to show that by suitable

choice of the constants F and k, which are still at our disposal, we can also satisfy conditions (2) and (3).

In polar co-ordinates ($x = r\cos\theta$),

$$\phi_1 = -Fr\cos\theta,$$
$$\phi_2 = -E_0\cos\theta\left(r - \frac{k}{r^2}\right).$$

The boundary conditions require

$$\phi_1 = \phi_2 \quad \text{and} \quad K_1\frac{\partial\phi_1}{\partial r} = K_2\frac{\partial\phi_2}{\partial r}, \text{ when } r = a.$$

We have therefore the two equations

$$F = E_0\left(1 - \frac{k}{a^3}\right),$$
$$K_1F = K_2E_0\left(1 + \frac{2k}{a^3}\right);$$

which give

$$F = E_0\frac{3K_2}{2K_2 + K_1} = E_0\left(1 - \frac{K_1 - K_2}{2K_2 + K_1}\right),$$
$$\frac{k}{a^3} = \frac{K_1 - K_2}{2K_2 + K_1}.$$

Consider the special case $K_2 = 1$, i.e. a *dielectric sphere in free space.* The field within it is weakened in the ratio $3 : 2 + K_1$, compared with the homogeneous field. It acts in space outside it like a dipole of moment

$$\mathbf{M} = \mathbf{E}_0k = \mathbf{E}_0a^3\frac{K_1 - 1}{K_1 + 2}. \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

Its polarization $\mathbf{P}$ (moment per unit volume) is

$$\mathbf{P} = \mathbf{E}_0\frac{3}{4\pi}\,\frac{K_1 - 1}{K_1 + 2}.$$

Comparison with the results already found for a conducting sphere (§ 8, p. 67) shows that the latter in its disturbing action on a homogeneous field behaves like an insulator of infinitely great dielectric constant.

CHAPTER V

Energy and Mechanical Forces in the Electrostatic Field

1. Charges and Metallic Conductors in Free Space.

The difference in point of view between Maxwell's field theory and the older theory of action at a distance comes out in a very significant way when they are applied to the following problem. Let it be required to find the work necessary to produce a given electrostatic distribution of charge, on the supposition that the charges are initially at infinitely great distances from each other and are brought up in succession to their final positions. We shall first take the case of individual point charges e_1, e_2, e_3, . . ., which are to be brought from infinity into positions where their mutual distances are r_{12}, r_{13}, &c. We first bring up the charge e_1 to its right place. This requires no work, since the rest of the charges are all at an infinite distance. Now bring up the charge e_2 to the distance r_{12} from e_1. This requires, on account of the Coulomb repulsion, work of amount

$$w_1 = \frac{e_1 e_2}{r_{12}}.$$

Next, we bring forward e_3, and for this have to expend the work

$$w_2 = \frac{e_1 e_3}{r_{13}} + \frac{e_2 e_3}{r_{23}}.$$

We proceed in this way until the whole of the n charges have been brought up to their places. Thus the total amount of work is

$$W = w_1 + w_2 + \ldots + w_n,$$

which may be written in the form

$$W = \tfrac{1}{2} e_1 \left(\frac{e_2}{r_{12}} + \frac{e_3}{r_{13}} + \ldots + \frac{e_n}{r_{1n}} \right) + \tfrac{1}{2} e_2 \left(\frac{e_1}{r_{21}} + \frac{e_3}{r_{23}} + \ldots + \frac{e_n}{r_{2n}} \right) + \ldots;$$

$$W = \tfrac{1}{2} \sum_{i=1}^{n} e_i \Phi_i, \quad \ldots \ldots \quad (1)$$

where Φ_i is the potential produced at the position of the ith charge by the *rest* of the charges.

Equation (1) is in a form which suits the action at a distance view. If we put a question about the localization of the energy expended, the answer will be that the energy in potential form is attached to the individual charges, the share which falls to the ith charge being $\frac{1}{2}e_i\Phi_i$. Quite a different answer is given by Maxwell's theory, which asserts that it is the field which carries the electrical energy, in accordance with the theorem: every element of volume dV of empty space, in which there is an electric field, contains a quantity of energy equal to

$$\frac{1}{8\pi}\mathbf{E}^2 dV.$$

A finite volume therefore contains a quantity of energy U, where

$$U = \iiint \frac{1}{8\pi}\mathbf{E}^2 dV. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To justify the assertion of the truth of this theorem, we shall first prove that the work W of equation (1), i.e. the work expended in bringing the n point charges within finite distances of each other, is identical with the increase in the quantity U resulting from this process. To prove this, denote by $\mathbf{E}_1, \mathbf{E}_2, \ldots$ the intensities due to the charges $e_1, e_2, \ldots$ at an arbitrary point of the field. Then we have $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \ldots + \mathbf{E}_n$, so that

$$\begin{aligned}\mathbf{E}^2 = {} & \mathbf{E}_1^2 + \mathbf{E}_2^2 + \ldots + \mathbf{E}_n^2 \\ & + \mathbf{E}_1(\mathbf{E}_2 + \mathbf{E}_3 + \ldots + \mathbf{E}_n) \\ & + \mathbf{E}_2(\mathbf{E}_1 + \mathbf{E}_3 + \ldots + \mathbf{E}_n) \\ & + \ldots \\ & + \mathbf{E}_n(\mathbf{E}_1 + \mathbf{E}_2 + \ldots + \mathbf{E}_{n-1}).\end{aligned}$$

If from this expression we form the integral (2) for U, we see in the first place that the contributions from the first row do not change at all in consequence of the mutual approach of the charges. In fact,

$$\frac{1}{8\pi}\iiint \mathbf{E}_1^2 dV,$$

for example, is the energy of the first charge by itself alone. (It would correspond to the work which would have to be expended in order to condense the first charge into a small volume from its original condition as a cloud of infinitely small density. For an actual point charge it would in fact be infinite.) Hence we need consider only the mutual action terms which appear in the other rows. If we write

$$\mathbf{E}_2 + \ldots + \mathbf{E}_n = -\operatorname{grad}\Phi_1,$$

so that Φ_1 denotes the potential due to all the charges except the first, the second row in the expression for $\mathbf{E}^2$ contributes to U the term

$$-\frac{1}{8\pi}\iiint(\mathbf{E}_1 \operatorname{grad}\Phi_1)dV,$$

for which, by Green's theorem (p. 19), we can write

$$-\frac{1}{8\pi}\left\{\iint\Phi_1 E_{1n}dS-\iiint\Phi_1 \operatorname{div}\mathbf{E}_1 dV\right\}.$$

For the boundary S we choose the infinitely distant sphere, which contributes nothing, along with a small spherical surface enclosing the first charge. On the latter we may take Φ_1 to be constant; also the charge e_1 is given by

$$-\frac{1}{4\pi}\iint E_{1n}dS=e_1,$$

so that on the whole

$$\frac{1}{8\pi}\iiint\mathbf{E}_1(\mathbf{E}_2+\ldots+\mathbf{E}_n)dV=\tfrac{1}{2}e_1\Phi_1.$$

By a corresponding transformation of the remaining rows in the expression for $\mathbf{E}^2$, we find

$$U-U_0=\tfrac{1}{2}\sum_{i=1}^{i=n}e_i\Phi_i=\tfrac{1}{2}\sum_{i,k}\frac{e_i e_k}{r_{ik}},$$

so that $U-U_0$ is actually equal to the work W which is expended. We are therefore justified in attaching to the quantity U defined in (2) the name "field energy".

If we consider any displacement of the point charges into neighbouring positions, clearly we have also the theorem: *the work required to effect a displacement of any given point charges is equal to the resulting increase in the field energy.*

The work used up is stored as potential energy in the field.

Although the case of point charges is a little inconvenient as regards the calculations, on account of the field becoming infinite, we have put this case in the forefront because of its relation to the Coulomb law of force. If the charges are distributed continuously in space, with volume density ρ, the corresponding relation becomes formally much simpler.

Green's theorem

$$\iint\phi\frac{\partial\psi}{\partial n}dS=\iiint(\phi\Delta\psi+\operatorname{grad}\phi\operatorname{grad}\psi)dV$$

now gives at once, when we put $\psi=\phi$, $\Delta\phi=-4\pi\rho$, $\operatorname{grad}\phi=-\mathbf{E}$,

$$\tfrac{1}{2}\int\rho\phi\,dV=\frac{1}{8\pi}\int\mathbf{E}^2 dV.$$

Here on the left we have the energy localized in the elements of charge, the element situated at a place where the potential is ϕ being $\rho\, dV$. On the right the energy appears as distributed over the field with energy density $\mathbf{E}^2/8\pi$.

In the case of superficial distributions of charge on the surface of metallic conductors (surface density of charge $= \omega$), the field being otherwise free of charge, $\Delta\phi$ in Green's theorem is 0, but on the left the surface integral taken over the various conductors still remains, and we have

$$U = \frac{1}{8\pi}\int \mathbf{E}^2 dV = \frac{1}{8\pi}\int \phi \frac{\partial \phi}{\partial n} dS. \quad . \quad . \quad . \quad . \quad (3)$$

The surface S is composed of the surfaces S_1, S_2, . . . of the separate conductors, the potentials of which we suppose to have the constant values $\phi_1, \phi_2, \ldots$. But $\partial\phi/\partial n$ is equal to 4π (surface density), so that

$$\int_{S_k} \frac{\partial \phi}{\partial n} dS_k = 4\pi e_k,$$

where e_k denotes the charge on the kth conductor. For the energy of the system of conductors 1, 2, . . . we therefore obtain

$$U = \tfrac{1}{2}(\phi_1 e_1 + \phi_2 e_2 + \ldots).$$

The energy of a condenser with charges $+e$ and $-e$ on the opposite coatings is therefore

$$U = \tfrac{1}{2}e(\phi_1 - \phi_2) = \tfrac{1}{2}C(\phi_1 - \phi_2)^2 = \tfrac{1}{2}\frac{e^2}{C},$$

where, as before, $$C = \frac{e}{\phi_1 - \phi_2}$$

denotes the capacity of the condenser.

2. Energy of the Field when Insulators are Present.

The expression just obtained for the energy of a condenser

$$U = \tfrac{1}{2}e(\phi_1 - \phi_2) = \tfrac{1}{2}C(\phi_1 - \phi_2)^2$$

is obviously quite independent of the presence or absence of an insulator between the coatings, for it represents the work required to charge the condenser. In a plate condenser in a vacuum (area of each plate S, plate distance d) we have

$$e = \frac{1}{4\pi} S\,|\,\mathbf{E}\,| \quad \text{and} \quad \phi_1 - \phi_2 = |\,\mathbf{E}\,|\, d,$$

so that, in agreement with (2), p. 82,

$$U = \frac{1}{8\pi} Sd\,|\,\mathbf{E}\,|^2.$$

In a dielectric with the dielectric constant K, the surface density ω of the charge on one of the metal plates is defined, not by the normal component of $\mathbf{E}$, but by that of $\mathbf{D} = K\mathbf{E}$. Hence we now have

$$e = \frac{K}{4\pi} S \,|\, \mathbf{E} \,|$$

and the energy
$$U = Sd\,\frac{K}{8\pi} \,|\, \mathbf{E} \,|^2.$$

The formula for the *energy density* u *of the electric field* must therefore now run *

$$u = \frac{K}{8\pi} \,|\, \mathbf{E} \,|^2 = \frac{1}{8\pi}(\mathbf{DE}). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For an infinite, continuously varying field, since $\operatorname{div} \mathbf{D} = 4\pi\rho$ and $\mathbf{E} = -\operatorname{grad} \phi$, we actually have

$$\frac{1}{8\pi}\int \mathbf{DE}\,dV = -\frac{1}{8\pi}\int \mathbf{D} \operatorname{grad} \phi\, dV$$
$$= \frac{1}{8\pi}\int \phi \operatorname{div} \mathbf{D}\, dV = \tfrac{1}{2}\int \phi\rho\, dV.$$

But that is just the expression which, from the standpoint of the theory of action at a distance, we ought to expect. The justification of the expression for the energy density of the field assumed in (4) will form an essential part of the following sections. Let it be remarked at once, however, that the assumption reaches far beyond electrostatics, and in particular that it remains valid even for fields which vary with the time. The main point is that it gives us a general method of calculating the forces which occur in the electrostatic field. For this application, we start from the fundamental theorem that the work done by the field in an arbitrary displacement of the charges is equal to the loss of field energy. The field energy thus transformed can appear experimentally in the most various forms: either in the form of kinetic energy when the carrier of the charge is free to move (free electrons), or in the form of heat when the carrier moves against a resistance of a frictional nature, or in the form of mechanical potential energy when work is done in the motion of the carrier against an external conservative force (e.g. gravity or an elastic force).

In the simplest case, that of a *homogeneous dielectric*, the new form (4) for the energy density is not essentially more complicated than the

* Properly speaking, (4) gives, not the energy density, but the density of the free energy (in the sense of thermodynamics). We shall always in fact in the developments of the following sections confine ourselves to isothermal processes, without specially mentioning in each case that we are doing so. To justify the results obtained in the text additional considerations are therefore very necessary. These, however, we shall postpone till later (p. 232).

form for free space. Suppose that we have an arbitrary system of metallic bodies, which to begin with is in electrostatic equilibrium in a vacuum. Let e_n and ϕ_n be the charge and potential of the nth conductor. Let $\mathbf{E}_0$ be the intensity at any point of the field, $u_0 = \mathbf{E}_0^2/8\pi$ the energy density. Next, fill the space with a dielectric of dielectric constant K. What does (4) give for the energy density now? If $\mathbf{E}$ and $\mathbf{D}$ are the intensity and displacement in the new field, we have always $\mathbf{D} = K\mathbf{E}$. But we must make a distinction between the two cases when e_n and ϕ_n, respectively, are kept constant when the dielectric is introduced.

(*a*) *The charges* e_n *are kept constant* (by insulation of the individual conductors). We then have

$$\int D_n dS = \int E_{0n} dS.$$

The sources of $\mathbf{D}$ are the same as those of the original $\mathbf{E}_0$. But this implies $\mathbf{D} = \mathbf{E}_0$, so that we have $\mathbf{E} = \mathbf{E}_0/K$, and therefore

$$u = \frac{1}{8\pi}(\mathbf{ED}) = \frac{u_0}{K}.$$

Further, since ϕ is always equal to $\int \mathbf{E}\,ds$, it follows that the potential likewise falls to $(1/K)$ of its former value; and, since the force between two charges is always determined by the field energy, Coulomb's law must now run

$$|\mathbf{F}| = \frac{e_1 e_2}{K r_{12}^2}.$$

When a homogeneous dielectric is introduced so as to fill the space between insulated conductors (the charges on which are kept constant), the field energy, intensity, potentials, and mutual forces all fall to $1/K$ *of their original values.*

From the point of view of the principle of energy, the question at once suggests itself: where has the energy gone that is lost in this filling process? The fact is, the introduction of the dielectric itself provides the compensation, for energy can be derived from the process. Since, however, while the process of introducing the insulator is going on, the dielectric is not homogeneous (K is even discontinuous at the surface of the insulator), we cannot at present calculate the forces involved.

(*b*) *The potentials of the conductors are kept constant* (e.g. by connecting them with the poles of voltaic cells). The potentials ϕ (in a homogeneous dielectric with the constant K) define the intensities $\mathbf{E}$ uniquely. We have therefore in this case

$$\mathbf{E} = \mathbf{E}_0, \text{ and consequently } \mathbf{D} = K\mathbf{E}_0;$$

hence $$u = \frac{1}{8\pi}(\mathbf{ED}) = Ku_0,$$

and also $$e_1 = \frac{1}{4\pi}\int D_n dS = Ke_{10},$$

and so on.

When a homogeneous dielectric is introduced so as to fill the space between conductors, whose potentials are kept constant, the field energy, displacement, charges, and mutual forces all increase to K *times their original values.*

The question of the energy balance comes up here also. In the first place, as we have already seen, we gain energy by the introduction of the insulator. But besides this the field energy is increased. Both these amounts of energy must be provided by the cells or batteries which maintain constant potentials on the various conductors. In fact the charge on the ith conductor is increased by the amount $e_i(K-1)$. The battery concerned must therefore provide energy of amount

$$\phi_i e_i(K-1).$$

The energy supplied by the batteries amounts accordingly on the whole to

$$A = \sum_i \phi_i e_i(K-1) = 2(K-1)U_0,$$

where $$U_0 = \tfrac{1}{2}\Sigma\phi_i e_i = \int\int\int u_0 dV$$

denotes the energy of the system when placed in a vacuum, with potentials ϕ. The increase in the field energy amounts, however, to only half of A, viz. to

$$KU_0 - U_0 = (K-1)U_0.$$

Thus we see that when the dielectric is brought into the field, the energy of which is U_0, the conductors being kept at constant potentials, energy is first gained of amount $(K-1)U_0$, then secondly the field energy is increased by a like amount. The energy first gained and the increase in the field energy are both supplied by the sources of current whose function it is to keep the potentials constant.

3. Thomson's Theorem.

When charges move in an electrostatic field under the action of the field strength, the energy of the field diminishes by the amount of the work done. The charges will accordingly, so far as they are free to move, tend to arrange themselves in such a way that the field energy will have the least possible value. If in particular metallic conductors are given with charges which in the first instance are arbitrary, these charges will distribute themselves so that the field

energy becomes a minimum. On the other hand, we know that in electrostatic equilibrium the potential of a conductor is constant, and its whole charge resides on its surface; we may expect therefore that this distribution of charge actually corresponds to a minimum for the field energy. In verification of this conjecture we shall now prove the following theorem. We are given a system of metallic conductors, embedded in a dielectric, whose dielectric constant K is an arbitrary function of position. Discontinuities in K can be excluded, however, without loss of generality, since any surface where the value of K changes suddenly may be imagined to be replaced by a narrow region of continuous but very rapid change. The individual insulated conductors carry the charges e_1, e_2, &c.

Now let the corresponding electrostatic field be described by means of $\mathbf{D}$ and $\mathbf{E}$; i.e. let the following equations be satisfied:

$$\iint_{s_i} D_n dS = 4\pi e_i \quad (i = 1, 2, \ldots, n) \quad . \quad . \quad . \quad (\alpha)$$

for the conducting surfaces; and in the field

$$\operatorname{div} \mathbf{D} = 4\pi\rho, \; \mathbf{D} = K\mathbf{E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

where the charges e_i and the functions of position ρ and K are given and fixed. To these, which we shall call the "general" conditions, have to be added the special electrostatic conditions

$$\phi = \phi_i = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

on each separate conductor, as well as

$$\operatorname{curl} \mathbf{E} = 0, \text{ or } \mathbf{E} = -\operatorname{grad} \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad (\delta)$$

everywhere.

Next, let $\mathbf{D}'$ and $\mathbf{E}'$ be any other field, of which we only know that it satisfies the general conditions (α) and (β) and that it is different from $\mathbf{D}$ and $\mathbf{E}$. We assert that these assumed data are sufficient to allow us to infer that

$$U' > U,$$

where $$U' = \frac{1}{8\pi}\int (\mathbf{D}'\mathbf{E}')dV \text{ and } U = \frac{1}{8\pi}\int (\mathbf{DE})\,dV$$

are the energies of the two fields.

To prove our assertion, we put

$$\mathbf{E}' = \mathbf{E} + \mathbf{E}'', \; \mathbf{D}' = \mathbf{D} + \mathbf{D}''.$$

For the two fields so introduced we have, by (α) and (β),

$$\iint_{s_i} D_n''dS = 0; \; \operatorname{div} \mathbf{D}'' = 0; \; \mathbf{D}'' = K\mathbf{E}''. \quad . \quad . \quad (\epsilon)$$

Thus
$$U' = \frac{1}{8\pi}\iiint (\mathbf{E}+\mathbf{E}'')(\mathbf{D}+\mathbf{D}'')dV$$
$$= \frac{1}{8\pi}\int (\mathbf{ED})dV + \frac{1}{8\pi}\int (\mathbf{E}''\mathbf{D}'')dV$$
$$+ \frac{1}{8\pi}\int (\mathbf{ED}'' + \mathbf{E}''\mathbf{D})dV;$$

and therefore, since $\mathbf{D}'' = K\mathbf{E}''$,
$$U' = U + \frac{1}{8\pi}\int K\mathbf{E}''^2 dV + \frac{1}{4\pi}\int (\mathbf{ED}'')dV.$$

We now use hypothesis (δ) that $\mathbf{E}$ is irrotational:
$$(\mathbf{ED}'') = -(\operatorname{grad}\phi,\ \mathbf{D}'') = -\operatorname{div}(\phi\mathbf{D}'') + \phi \operatorname{div}(\mathbf{D}'').$$

By hypothesis (γ) $\mathbf{E}$ is nil everywhere within the metals. We can therefore, when calculating the volume integral
$$\frac{1}{4\pi}\int \mathbf{ED}''dV,$$

confine ourselves to the region occupied by dielectric, in which $\operatorname{div}\mathbf{D}'' = 0$. We thus find
$$\int \mathbf{ED}''dV = \sum_i \int_{S_i} \phi \mathbf{D}_n''dS$$
$$= \sum_i \phi_i \int_{S_i} D_n''dS = 0,$$

since by (γ) ϕ is constant on each conductor. This proves Thomson's theorem. We have in fact
$$U' = U + \frac{1}{8\pi}\int K\mathbf{E}''^2 dV.$$

U' is therefore greater than U, provided only that at some part of the space $\mathbf{E}'$ is different from $\mathbf{E}$.

Thomson's theorem therefore deduces the field which corresponds to the equilibrium distribution of electricity, from a minimum principle. This minimum principle corresponds exactly to the condition of equilibrium which holds for heavy bodies in the field of gravity; such bodies are in equilibrium, in stable equilibrium in fact, when the potential energy of gravity has its smallest value for the configuration in question. In the same way we see here that the equilibrium of electricity on the surface of fixed conductors is characterized by a minimum value of the electrical energy. *Electrical energy accordingly plays the same part here as potential energy does in ordinary mechanics.*

The foregoing developments show once again that there cannot

be two different solutions of the problem of electrostatics. For the inequality we have proved states that, for every field $\mathbf{E}'$, $\mathbf{D}'$, which is different from the electrostatic one, but which satisfies the conditions (α) and (β), the energy U' is greater than U. If now $\mathbf{E}'$, $\mathbf{D}'$ is itself an electrostatic field, we must also have $U > U'$, which is impossible. Hence there cannot be two different solutions of the problem of electrostatics; *conditions* (α) *to* (δ) *define the electrostatic field uniquely.*

4. Dielectric Sphere in a Non-homogeneous Field.

Before proceeding in next section to deduce the most general expression for the ponderomotive forces from the principle of energy, we shall here calculate in a more direct way the force which a charged or uncharged sphere is subjected to in a non-homogeneous field. We shall first, however, consider the case of a large number—a "cloud"—of point charges $e_1, e_2, \ldots, e_h$ situated at the points $(x_1, y_1, z_1), \ldots, (x_h, y_h, z_h)$ in the neighbourhood of the origin of co-ordinates. The dimensions of the cloud are supposed so small that we can represent the external field with sufficient approximation by

$$E_x = E_{x0} + \left(\frac{\partial E_x}{\partial x}\right)_0 x + \left(\frac{\partial E_x}{\partial y}\right)_0 y + \left(\frac{\partial E_x}{\partial z}\right)_0 z,$$

with similar expressions for E_y and E_z; or, more briefly, by

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 + (\mathbf{r}\,\mathrm{grad}_0)\,\mathbf{E},$$

where the suffix 0 indicates that the value of the quantity in question is to be taken at the origin. The force acting on our cloud is now

$$\begin{aligned}\mathbf{F} &= \sum_{i=1}^{h} e_i \mathbf{E}(\mathbf{r}_i)\\ &= \mathbf{E}_0 \sum_{i=1}^{i=h} e_i + (\Sigma e_i \mathbf{r}_i\,\mathrm{grad})\,\mathbf{E}.\end{aligned}$$

Now
$$\sum_{i=1}^{h} e_i \mathbf{r}_i = \mathbf{M}$$

is the *electric moment* of our cloud of charges (cf. § 5, p. 22). If $\Sigma e_i = e$ is its total charge, we find

$$\mathbf{F} = \mathbf{E}_0 e + (\mathbf{M}\,\mathrm{grad})\,\mathbf{E}.$$

We apply this result to a sphere of dielectric constant K and radius a. By equation (6a), p. 80, this becomes polarized in a homogeneous field $\mathbf{E}$ in such a way that its polarization charges act in the space outside like a dipole of moment

$$\mathbf{M} = a^3 \frac{K-1}{K+2} \mathbf{E}_0.$$

If the dimensions of the sphere are not too great, this expression will continue to be approximately correct even in a non-homogeneous field. If in addition the sphere carries a charge e, the total force acting on it is

$$\mathbf{F} = e\mathbf{E} + a^3 \frac{K-1}{K+2} (\mathbf{E}\,\mathrm{grad})\,\mathbf{E}.$$

In the electrostatic field we have, since $\mathbf{E}$ is irrotational,

$$(\mathbf{E}\,\mathrm{grad})\,\mathbf{E} = \tfrac{1}{2}\,\mathrm{grad}\,\mathbf{E}^2,$$

as may easily be proved by writing out the x-component, for example.

We have therefore

$$\mathbf{F} = e\mathbf{E} + \tfrac{1}{2} a^3 \frac{K-1}{K+2}\,\mathrm{grad}\,\mathbf{E}^2. \quad . \quad . \quad . \quad . \quad (4a)$$

The uncharged sphere is impelled towards places of higher field strength. If in particular we pass to the limit $K \rightarrow \infty$, the force acting upon a metallic uncharged sphere of radius a becomes

$$\mathbf{F} = \tfrac{1}{2} a^3\,\mathrm{grad}\,\mathbf{E}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

It follows that $\frac{1}{2}a^3\mathbf{E}^2$ is the energy which must be expended to withdraw a metallic sphere of radius a from a field $\mathbf{E}$ into a region where there is no field.

We are now in a position to assign the conditions which a testing body (proof body) must satisfy before it can be used to measure a field by means of the simple relation $\mathbf{F} = e\mathbf{E}$. In the first place the radius a of the sphere must be so small that in (4a) the second term can be neglected in comparison with the first. Secondly, the charge e itself must be so small that the forces exerted by uncharged metal surfaces and by the surfaces of insulators, which may be represented as due to images (cf. § 7, p. 65) and are proportional to e^2, are negligible compared with $e\mathbf{E}$. The nearer we approach a disturbing surface, and the farther removed the field is from homogeneity, the smaller will the charge and diameter of the proof body have to be.

5. Mechanical Forces in the Electrostatic Field.

We shall now deduce the general expression for the force $\mathfrak{f}dV$ which acts upon a particle occupying the element of volume dV in the field. For this purpose we employ the principle of the conservation of energy as follows. Think of the matter in the field as moving in any way, and let $\mathbf{u}(x, y, z)$ be the current vector which defines the velocity of the particle situated at the point (x, y, z). We shall assume that $\mathbf{u}$ is sufficiently small in absolute value to allow us to regard the field at any given moment as electrostatic. (The following considerations therefore apply strictly only in the limit $|\,\mathbf{u}\,| \rightarrow 0$.) The scalar

product $(\mathbf{uf})dV$ clearly gives the work done per second in this motion by the force-density $\mathbf{f}$ on the element of volume dV. If we denote the whole energy of the field by

$$U = \frac{1}{8\pi}\int(\mathbf{ED})dV, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the principle of energy requires that the rate of decrease of U per second should be equal to the total work done on the matter:

$$\frac{dU}{dt} = -\int(\mathbf{uf})dV. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In point of fact we do not know $\mathbf{f}$ to begin with; the real problem before us is to transform the time rate of change of the function U defined by (5), so that it may assume the form (6). When this has been done, we shall be justified in regarding the factor $\mathbf{f}$ so found as a force-density compatible with the principle of energy. We have therefore to calculate how U changes in consequence of the motion $\mathbf{u}$.

Now the field is uniquely defined when the charge density ρ and the dielectric constant K are everywhere given. Hence U only varies with the time in so far as ρ and K vary. We therefore calculate first the two expressions $\delta_\rho U$ and $\delta_K U$. Here $\delta_\rho U$ denotes the change in U consequent on a change of $\rho(x, y, z)$ to $\rho + \delta\rho(x, y, z)$ while $K(x, y, z)$ is kept constant, and similarly $\delta_K U$ denotes the change in U when ρ is kept constant and K varies.

In each case we have an energy difference between two fields to calculate:

$$\delta U = U_2 - U_1 = \frac{1}{8\pi}\int(\mathbf{E}_2\mathbf{D}_2 - \mathbf{E}_1\mathbf{D}_1)dV. \quad . \quad . \quad (7)$$

(*a*) *When K is kept constant*, we have $\mathbf{E}_1\mathbf{D}_2 = \mathbf{E}_2\mathbf{D}_1$, and therefore

$$(\mathbf{E}_2\mathbf{D}_2 - \mathbf{E}_1\mathbf{D}_1) = (\mathbf{E}_1 + \mathbf{E}_2)(\mathbf{D}_2 - \mathbf{D}_1).$$

Since we are confining ourselves to small changes, this gives

$$\delta_\rho U = \frac{1}{4\pi}\int\mathbf{E}\,\delta\mathbf{D}\,dV.$$

Now $\quad \mathbf{E}\,\delta\mathbf{D} = -(\text{grad}\,\phi,\ \delta\mathbf{D}) = -\text{div}(\phi\,\delta\mathbf{D}) + \phi\ \text{div}(\delta\mathbf{D}).$

But the change in the charge density $\delta\rho$ is equivalent to a change in the divergence of the displacement vector,

$$4\pi\,\delta\rho = \text{div}\,\delta\mathbf{D}.$$

Hence

$$\delta_\rho U = \int\phi\,\delta\rho\,dV. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This result includes as a special case the theorem that in a small dis-

placement of the charges on an insulated conductor the field energy is not changed; in fact in the state of electrostatic equilibrium thus specified ϕ is constant, and $\int \delta\rho\, dV = 0$. This of course is simply a special case of Thomson's theorem, that the electrostatic energy is a minimum. This special case is of importance here for this reason, that it shows that the part of $\delta\rho$, which consists of a motion of the charges on metallic conductors, contributes nothing to the time rate of change of U, so long as electrostatic equilibrium continues to be maintained—which we have postulated to be the case.

(b) *When the density of charge ρ is kept constant,* $\mathbf{D}_2 - \mathbf{D}_1$ is everywhere solenoidal; further, $\mathbf{E}_1$ and $\mathbf{E}_2$ are irrotational, and we therefore have in this case (p. 39)

$$\int \mathbf{E}_1(\mathbf{D}_2 - \mathbf{D}_1)\, dV = 0,$$

and

$$\int \mathbf{E}_2(\mathbf{D}_2 - \mathbf{D}_1)\, dV = 0,$$

if both integrals are taken over the whole of space. We may therefore in (7) replace under the integral sign $\mathbf{E}_2\mathbf{D}_2$ by $\mathbf{E}_2\mathbf{D}_1$, and $\mathbf{E}_1\mathbf{D}_1$ by $\mathbf{E}_1\mathbf{D}_2$, so that we obtain

$$\delta_K U = \frac{1}{8\pi}\int (\mathbf{E}_2\mathbf{D}_1 - \mathbf{E}_1\mathbf{D}_2)\, dV.$$

But if K_1 and K_2 are the values of the dielectric constant before and after the change, we have $\mathbf{D}_1 = K_1\mathbf{E}_1$, and $\mathbf{D}_2 = K_2\mathbf{E}_2$, so that

$$\delta_K U = -\frac{1}{8\pi}\int (K_2 - K_1)\mathbf{E}_1\mathbf{E}_2\, dV.$$

For a small change δK this becomes

$$\delta_K U = -\frac{1}{8\pi}\int \delta K\, \mathbf{E}^2\, dV. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For the total rate of change of the field energy we have therefore

$$\frac{dU}{\partial t} = \int \phi \frac{\partial \rho}{\partial t}\, dV - \frac{1}{8\pi}\int \mathbf{E}^2 \frac{\partial K}{\partial t}\, dV. \quad . \quad . \quad . \quad (10)$$

We have now to bring the time rates of change of ρ and K into connexion with the prescribed rate of flow of the matter $\mathbf{u}$. With respect to $\partial\rho/\partial t$, by Thomson's theorem we need not take into account the motion of the electricity on a conducting surface. We may therefore make the calculations as if the charge were everywhere rigidly fixed to the matter. In that case, however, $\rho\mathbf{u}$ is the density of the convection current, so that by Gauss's theorem

$$\frac{\partial \rho}{\partial t} = -\mathrm{div}\,(\rho\mathbf{u}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

We note in passing the exactly similar formula for the rate of change of the mass-density σ of the moving substance,

$$\frac{\partial \sigma}{\partial t} = -\operatorname{div}(\mathbf{u}\sigma).$$

To calculate $\partial K/\partial t$ we consider the rate of change of K for a moving material particle. For this, we have to compare the value of K at time 0 at the point (x, y, z) with the corresponding value at time dt at the point $x + u_x dt$, $y + u_y dt$, $z + u_z dt$. For this rate of change for the material, or " substantial rate of change ", we obtain

$$\frac{dK}{dt} = \frac{K(dt,\ x + u_x dt,\ y + u_y dt,\ z + u_z dt) - K(0, x, y, z)}{dt}$$

$$= \frac{\partial K}{\partial t} + (\mathbf{u} \operatorname{grad} K),$$

and therefore
$$\frac{\partial K}{\partial t} = -(\mathbf{u} \operatorname{grad} K) + \frac{dK}{dt}.$$

The substantial rate of change dK/dt cannot be determined generally without further assumptions as to the nature of the dielectric. We restrict ourselves to the assumption—certainly sufficient for liquids—that K *is a single-valued function of the density* σ. It then follows that

$$\frac{dK}{dt} = \frac{dK}{d\sigma}\frac{d\sigma}{dt}.$$

Again, the substantial rate of change of the density is given generally by

$$\frac{d\sigma}{dt} = \frac{\partial \sigma}{\partial t} + (\mathbf{u} \operatorname{grad} \sigma),$$

and therefore, with the above noted value of $\partial\sigma/\partial t$,

$$\frac{d\sigma}{dt} = -\operatorname{div}(\mathbf{u}\sigma) + (\mathbf{u} \operatorname{grad} \sigma)$$

or
$$\frac{d\sigma}{dt} = -\sigma \operatorname{div} \mathbf{u}.$$

Thus we have finally

$$\frac{\partial K}{\partial t} = -\left\{(\mathbf{u} \operatorname{grad} K) + \sigma \frac{dK}{d\sigma} \operatorname{div} \mathbf{u}\right\}. \quad . \quad . \quad . \quad (12)$$

Using (11) and (12) in (10), we now have

$$\frac{dU}{dt} = -\int \phi \operatorname{div}(\rho \mathbf{u}) dV$$

$$+ \frac{1}{8\pi}\int \mathbf{E}^2\left(\mathbf{u} \operatorname{grad} K + \sigma \frac{dK}{d\sigma} \operatorname{div} \mathbf{u}\right) dV.$$

We next apply Gauss's theorem twice, after putting

$$\phi \operatorname{div} (\rho \mathbf{u}) = \operatorname{div} (\phi \rho \mathbf{u}) - (\rho \mathbf{u} \operatorname{grad} \phi)$$

and
$$\mathbf{E}^2 \sigma \frac{dK}{d\sigma} \operatorname{div} \mathbf{u} = \operatorname{div} \left(\mathbf{E}^2 \sigma \frac{dK}{d\sigma} \mathbf{u}\right) - \mathbf{u} \operatorname{grad} \left(\mathbf{E}^2 \frac{dK}{d\sigma} \sigma\right).$$

The integrals over the infinitely distant surface vanish, and there remains, since $\mathbf{u}$ now occurs as a factor in every term,

$$\frac{dU}{dt} = -\int \left\{\mathbf{u},\ \rho \mathbf{E} - \frac{1}{8\pi} \mathbf{E}^2 \operatorname{grad} K + \frac{1}{8\pi} \operatorname{grad} \left(\mathbf{E}^2 \frac{dK}{d\sigma} \sigma\right)\right\} dV.$$

We have thus actually obtained an expression of the form (6), from which we can at once deduce the force density $\mathbf{f}$:

$$\mathbf{f}_e = \rho \mathbf{E} - \frac{1}{8\pi} \mathbf{E}^2 \operatorname{grad} K + \frac{1}{8\pi} \operatorname{grad} \left(\mathbf{E}^2 \frac{dK}{d\sigma} \sigma\right). \quad . \quad (13)$$

Here, let us emphasize once again, it is assumed that the dielectric constant K is a function of the density σ alone.

The expression for $\mathbf{f}_e$ in (13) consists of three terms. The first, $\rho \mathbf{E}$, gives the known force on the true charges. The second,

$$-\frac{1}{8\pi} \mathbf{E}^2 \operatorname{grad} K,$$

makes a contribution to the value of $\mathbf{f}_e$, wherever K is variable from point to point. In particular, at the interface between an insulator and free space it gives a force at right angles to the surface of the insulator, tending to pull the insulator into the vacuum.

Lastly, the third term is important for the phenomenon of electrostriction, which we shall consider separately in next section.

6. Electrostriction in Chemically Homogeneous Liquids and Gases.

If an electrical field is excited within an uncharged dielectric, the first effect of the force (13) will be to cause relative motion of the parts of the dielectric medium. This will give rise to elastic forces opposing the motion. The motion ceases when the elastic forces exactly balance the electric force (13).

The phenomenon of the production of stresses and strains in this way in an uncharged insulator is called electrostriction.

In liquids and gases in equilibrium there is only one kind of elastic stress, namely equal pressure in all directions. In this case we may expect the circumstances to be particularly simple. In fact, if the pressure varies with position, a force due to the pressure gradient acts on an element of volume dV, viz.

$$\mathbf{f}_p dV = -\operatorname{grad} p \,.\, dV. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

There will be equilibrium when the total force on the element of volume, i.e. the pressure $\mathbf{f}_p$ along with the electrical force $\mathbf{f}_e$, is nil. The condition of equilibrium, in an uncharged insulator ($\rho = 0$), is therefore

$$-\frac{1}{8\pi}\mathbf{E}^2 \operatorname{grad} K + \frac{1}{8\pi}\operatorname{grad}\left(\mathbf{E}^2 \frac{dK}{d\sigma}\sigma\right) - \operatorname{grad} p = 0. \quad (14a)$$

If p and K are known as functions of the density σ, (14a) expresses a relation between the density (or the pressure) and the square of the field intensity.

Equation (14a) for the equilibrium pressure p in an uncharged liquid dielectric can be integrated generally, provided the functions $p = p(\sigma)$ and $K = K(\sigma)$ are known. For this purpose we form the gradient of

$$\mathbf{E}^2 \frac{dK}{d\sigma} = \mathbf{E}^2 \frac{dK}{d\sigma} \cdot \sigma \cdot \frac{1}{\sigma}$$

by the rule for the differentiation of a product:

$$\operatorname{grad}\left(\mathbf{E}^2 \frac{dK}{d\sigma}\right) = \frac{1}{\sigma}\operatorname{grad}\left(\mathbf{E}^2 \frac{dK}{d\sigma}\sigma\right) + \mathbf{E}^2 \frac{dK}{d\sigma}\sigma \operatorname{grad}\frac{1}{\sigma}.$$

Now obviously $\operatorname{grad} K = \frac{dK}{d\sigma}\operatorname{grad}\sigma$, and $\operatorname{grad}\frac{1}{\sigma} = -\frac{1}{\sigma^2}\operatorname{grad}\sigma$, so that $\frac{dK}{d\sigma}\sigma\operatorname{grad}\frac{1}{\sigma} = -\frac{1}{\sigma}\operatorname{grad} K$.

Hence $$\operatorname{grad}\left(\mathbf{E}^2 \frac{dK}{d\sigma}\right) = \frac{1}{\sigma}\left\{\operatorname{grad}\left(\mathbf{E}^2 \frac{dK}{d\sigma}\sigma\right) - \mathbf{E}^2 \operatorname{grad} K\right\}.$$

This gives from (14a)

$$\frac{1}{\sigma}\operatorname{grad} p = \operatorname{grad}\left(\frac{1}{8\pi}\mathbf{E}^2 \frac{dK}{d\sigma}\right). \quad . \quad . \quad . \quad . \quad (14b)$$

We now consider p on the left to be expressed as a function of σ, and form the integral

$$\psi(\sigma) = \int_{\sigma_0}^{\sigma} \frac{1}{\sigma}\frac{dp}{d\sigma}\,d\sigma,$$

where the lower limit σ_0 is fixed. Then

$$\operatorname{grad}\psi(\sigma) = \frac{d\psi}{d\sigma}\operatorname{grad}\sigma = \frac{1}{\sigma}\frac{dp}{d\sigma}\operatorname{grad}\sigma$$

$$= \frac{1}{\sigma}\operatorname{grad} p.$$

We have therefore the result: the gradient of

$$\psi(\sigma) - \frac{1}{8\pi} \mathbf{E}^2 \frac{dK}{d\sigma}$$

is nil; this quantity has accordingly a constant value throughout the dielectric. If we compare any two points in the fluid, distinguished by the indices 0 and 1, then, since ψ can also be considered as a function of the pressure, say

$$\psi(p) = \int_{p_0}^{p} \frac{dp}{\sigma(p)},$$

we have $$\int_{p_0}^{p_1} \frac{dp}{\sigma} = \frac{1}{8\pi} \left[\mathbf{E}_1^2 \cdot \left(\frac{dK}{d\sigma}\right)_{\sigma=\sigma_1} - \mathbf{E}_0^2 \cdot \left(\frac{dK}{d\sigma}\right)_{\sigma=\sigma_0} \right]. \qquad (14c)$$

If the absolute values of the intensities, $\mathbf{E}_1$ *and* $\mathbf{E}_0$, *are known,* (14*c*) *gives a relation between the densities* σ_1 *and* σ_0, *or between the pressures* p_1 *and* p_0.

We shall discuss (14*c*) for the case when $\mathbf{E}_0 = 0$; comparing e.g. the pressure in the dielectric between two condenser plates (intensity $|\mathbf{E}| = E_1$) with that outside the field:

$$\int_{p_0}^{p_1} \frac{dp}{\sigma} = \frac{1}{8\pi} E_1^2 \cdot \left(\frac{dK}{d\sigma}\right)_{\sigma=\sigma_1}.$$

If the dielectric is a *nearly incompressible liquid*, the density σ can be regarded as approximately constant. We then have

$$p_1 - p_0 = \frac{1}{8\pi} E_1^2 \sigma \frac{dK}{d\sigma}.$$

The result becomes even more explicit if we make use of the experimentally well-confirmed formula of Clausius-Mosotti, which gives

$$\frac{K-1}{K+2} = C\sigma, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14d)$$

C being a constant independent of the density σ. From this by differentiation we find

$$\frac{3}{(K+2)^2} dK = C\, d\sigma,$$

so that
$$\sigma \frac{dK}{d\sigma} = \sigma C \frac{(K+2)^2}{3}$$
$$= \tfrac{1}{3}(K+2)(K-1). \quad . \quad . \quad . \quad (14e)$$

For electrostriction in liquids we therefore have

$$p_1 - p_0 = \tfrac{1}{2} E_1^2 \frac{K+2}{3} \frac{K-1}{4\pi}.$$

For a not too dense gas, on the other hand, we can use the gas equation for p,

$$p = \sigma \frac{RT}{M}$$

(M = molecular weight), while the susceptibility is proportional to the density, or

$$K = 1 + \kappa\sigma, \quad \text{so that} \quad \frac{dK}{d\sigma} = \kappa = \frac{K-1}{\sigma}.$$

Hence, from (14c),

$$\frac{RT}{M} \log \frac{p_1}{p_0} = \tfrac{1}{2} E^2 \frac{K-1}{4\pi} \frac{1}{\sigma}.$$

If we denote by α the polarizability (see (1), p. 74) of a single molecule, i.e. the electric moment induced in it by the field intensity 1, and by n the number of molecules per cubic centimetre, then

$$\frac{K-1}{4\pi} = \alpha n, \quad \text{and} \quad \sigma = n \frac{M}{N},$$

where $N = 6 \times 10^{23}$ is the number of molecules in a gramme-molecule. Introducing Boltzmann's constant

$$k = \frac{R}{N} = 1{\cdot}37 \times 10^{-16}$$

(the gas constant for a single molecule) we find for gases

$$\frac{p_1}{p_0} = e^{\frac{1}{2}\alpha E^2/kT}. \qquad \qquad (14f)$$

We could have deduced this formula, however, by a quite different method directly from previous results regarding the force acting upon a dielectric sphere (equation (4a), p. 91). Thus, if we consider the single molecule as a sphere of dielectric or conducting material, with radius a, we have, by equation (6a), p. 80,

$$\alpha = a^3 \frac{K-1}{K+2}.$$

The force acting on this sphere in the non-homogeneous field is therefore

$$\mathbf{F} = \tfrac{1}{2}\alpha \operatorname{grad} \mathbf{E}^2 = \operatorname{grad} (\tfrac{1}{2}\alpha \mathbf{E}^2),$$

which can be derived from the potential $(-\frac{1}{2}\alpha\mathbf{E}^2)$. Thus (14f) is none other than the known barometric height formula

$$\frac{p}{p_0} = e^{-(mgh)/kT},$$

except only that the potential energy of a particle in the earth's field (mgh) is replaced by the corresponding energy in our non-homogeneous field ($-\frac{1}{2}\alpha\mathbf{E}^2$).

The rough model of a molecule just referred to allows us to take the further step of estimating the order of magnitude of the electrostriction (14f). In the first place, we get an order of magnitude for α which agrees with experiment, if we put $K = \infty$ (conducting sphere) and for a take 10^{-8} cm., the actual diameter of a molecule. Thus, in a field of 300,000 volts per cm. corresponding to $|\mathbf{E}| = 1000$, with $k = 1{\cdot}37 \times 10^{-16}$ and $T = 300$ e.g., we get

$$\frac{\frac{1}{2}\alpha\mathbf{E}^2}{kT} = \frac{10^{-24} \times 10^6}{2 \times 1{\cdot}37 \times 10^{-16} \times 300} \approx 10^{-5}.$$

The effect in question is therefore always very minute, and great care would be required to measure it.

We shall also discuss briefly the more general question of what conditions must be satisfied in order that a fluid should be in equilibrium under the action of the force given by (13), p. 95. The unique system of applied force which will maintain a fluid in equilibrium is the body force derived from the hydrostatic pressure p as in equation (14), p. 95, $\mathbf{f}_p = -\text{grad } p$. The condition of equilibrium is therefore

$$\mathbf{f}_e + \mathbf{f}_p = \mathbf{f}_e - \text{grad } p = 0.$$

The necessary and sufficient condition that the electric force $\mathbf{f}_e$ should be in equilibrium with a hydrostatic pressure gradient is therefore

$$\mathbf{f}_e = \text{grad } p,$$

or

$$\text{curl } \mathbf{f}_e = 0;$$

for every irrotational vector can be represented as the gradient of a scalar. Of the three terms of which $\mathbf{f}_e$ is composed, the third is always irrotational. The first two give the equation of equilibrium

$$\text{curl} \left\{ \rho\mathbf{E} - \frac{1}{8\pi} \mathbf{E}^2 \text{ grad } K \right\} = 0.$$

If we assume that the charge density ρ is zero except where the dielectric constant K has a constant value in space, then by the general vector formula

$$\text{curl } (\psi\mathbf{A}) = \psi \text{ curl } \mathbf{A} + [\text{grad } \psi, \mathbf{A}],$$

noting that $\mathbf{E} = -\text{grad } \phi$ is irrotational, we obtain the two conditions

$$\text{curl } (\rho\mathbf{E}) = -[\text{grad } \rho, \text{grad } \phi] = 0,$$

and

$$\text{curl } (\mathbf{E}^2 \text{ grad } K) = [\text{grad } \mathbf{E}^2, \text{grad } K] = 0.$$

The latter condition has been used already (at (14*a*)) in our discussion of electrostriction. It states that the surfaces $\mathbf{E}^2 =$ const. and $K =$ const. coincide. For it is only then that the gradients of the two quantities are parallel, as is requisite for the vanishing of their vector product. But since we have stipulated that K is to depend on the density only, this condition implies that K—and with it the density σ—must be a function of $\mathbf{E}^2$ alone; which is in agreement with our foregoing treatment of electrostriction.

In an exactly similar way the first equation implies that the charge density ρ must be a function of the electrostatic potential alone. This case is of practical importance in connexion with the behaviour of an electrolyte in contact with a metal electrode; in the neighbourhood of the metal electrode there is in general a fall of potential associated with a space charge of the electrolyte (excess of one kind of ion over another). Here the space charge does in point of fact depend only on the potential at the point concerned.

7. The Mechanical Force at the Surface of a Dielectric.

According to equation (14*c*) of the preceding section, the hydrostatic pressure p at any point in the interior of an uncharged dielectric can be calculated, provided the equation of state $p = p(\sigma)$ of the dielectric for the case of no field, and also the law of dependence of the dielectric constant K on the density σ, are known. With regard to the application of this equation, the following circumstance must be emphasized. The *immediate* experimental significance of the pressure difference given by (14*c*) consists in this *only*, that it can be used to calculate the changes of density in the electric field, say for example the quantity of fluid absorbed by a condenser. On the other hand, the pressure calculated by (14*c*) tells us nothing in the first instance about the force which the dielectric exerts upon the walls of the vessel in which it is enclosed. The very steep gradient in the dielectric constant which from the nature of the case occurs at the surface is accompanied, according to the general formula (14*a*), by a correspondingly steep gradient in the pressure p. To illustrate this, we consider the following arrangement (fig. 1).

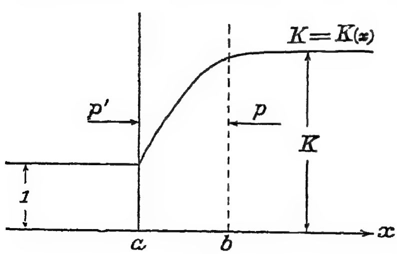

Fig. 1

We take the x-axis of a co-ordinate system at right angles to the surface of the fluid, and replace the instantaneous jump in the value of K by a change which, though very rapid, is continuous. At $x = a$, K is to have the value for a vacuum, viz. 1; from $x = b$ onwards

we are in the interior of the dielectric. At $x = a$ a piston exerts on the dielectric from the left a pressure p' per unit area. The material of which the piston itself is made possesses likewise the dielectric constant 1. To the layer of fluid between a and b we now apply the general equation of equilibrium (14a). This states that between a and b a pressure difference $p' - p$ must operate, which exactly compensates for the electrical body-force acting upon the layer of fluid. Since it is obviously only the x-component of the force which matters for our purpose, (14a) runs in our case:

$$p' - p = \frac{1}{8\pi}\int_a^b \mathbf{E}^2 \frac{\partial K}{\partial x}\,dx - \frac{1}{8\pi}\int_a^b \frac{\partial}{\partial x}\left(\mathbf{E}^2 \frac{dK}{d\sigma}\sigma\right)dx.$$

On the left here we have the pressure difference acting from left to right, on the right the electrical body-force acting from right to left.

Since σ is zero at $x = a$ by hypothesis, we have

$$p' - p = \frac{1}{8\pi}\int_a^b \mathbf{E}^2 \frac{\partial K}{\partial x}\,dx - \left(\frac{1}{8\pi}\mathbf{E}^2 \frac{dK}{d\sigma}\sigma\right)_{x=b}. \quad . \quad (15)$$

The integral still remaining in (15) can be evaluated generally by observing that the tangential component of $\mathbf{E}$ and the normal component of $K\mathbf{E}$ are continuous across the boundary. We denote the normal and tangential components by the indices n and t, so that we can write

$$\mathbf{E}^2 = E_t^2 + E_n^2.$$

We thus have

$$\int_a^b \mathbf{E}^2 \frac{\partial K}{\partial x}\,dx = \int_a^b E_t^2 \frac{\partial K}{\partial x}\,dx + \int_a^b (KE_n)^2 \frac{1}{K^2}\frac{\partial K}{\partial x}\,dx$$
$$= E_t^2(K_b - 1) + (KE_n)^2\left(1 - \frac{1}{K_b}\right).$$

Quantities written without an index, and in particular E_n, we shall agree to take as for $x = b$ *within* the dielectric. We shall then have

$$\int_a^b \mathbf{E}^2 \frac{\partial K}{\partial x}\,dx = (K-1)\{E_t^2 + KE_n^2\}, \quad . \quad . \quad (15')$$

or, after addition and subtraction of E_n^2 in the crooked brackets,

$$\int_a^b \mathbf{E}^2 \frac{\partial K}{\partial x}\,dx = (K-1)\mathbf{E}^2 + (K-1)^2 E_n^2.$$

Hence, from (15),

$$p' - p = \mathbf{E}^2 \frac{1}{8\pi}\left\{(K-1) - \frac{d(K-1)}{d\sigma}\sigma\right\} + E_n^2 \frac{(K-1)^2}{8\pi}. \quad (15a)$$

The pressure p$'$ *exerted by the fluid on a non-polarizable wall*

therefore exceeds by this amount the hydrostatic pressure p *within the fluid.*

Within the range of validity of the formula of Clausius-Mosotti (14*d*, *e*) we have

$$K-1-\frac{d(K-1)}{d\sigma}\sigma=(K-1)\left(1-\frac{K+2}{3}\right)=-\tfrac{1}{3}(K-1)^2.$$

Hence $$p'-p=\frac{(K-1)^2}{8\pi}\left(\tfrac{2}{3}E_n^2-\tfrac{1}{3}E_t^2\right). \quad . \quad . \quad . \quad (15b)$$

Thus p' is greater than p if $\mathbf{E}$ is at right angles to the boundary surface, but smaller than p if $\mathbf{E}$ lies in that surface.

We shall at a later stage give another proof of (15*a*) by a purely thermodynamical method. Meantime, we shall illustrate the meaning of the general relation (15) by two simple examples. For that purpose we write (15) in the form

$$p'=\frac{1}{8\pi}\int_a^b \mathbf{E}^2\frac{\partial K}{\partial x}dx+p-\frac{1}{8\pi}\mathbf{E}^2\frac{dK}{d\sigma}\sigma, \quad . \quad . \quad (15c)$$

and use the formula deduced above (p. 97) for slightly compressible fluids, according to which the quantity

$$p-\frac{1}{8\pi}\mathbf{E}^2\frac{dK}{d\sigma}\sigma$$

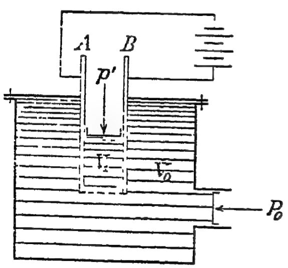

Fig. 2

has the same value everywhere within the fluid. If then (fig. 2) the plates A and B of a charged condenser dip into a liquid dielectric which, in those parts of it where there is no field, is under the pressure p_0, we have

$$p-\frac{1}{8\pi}\mathbf{E}^2\frac{dK}{d\sigma}\sigma=p_0. \quad . \quad . \quad (15d)$$

The force with which the liquid is drawn into the condenser can therefore be specified by the difference

$$p'-p_0=\frac{1}{8\pi}\int_a^b \mathbf{E}^2\frac{\partial K}{\partial x}dx. \quad . \quad . \quad . \quad . \quad (15e)$$

This result is often replaced by the statement that, at the surface of the liquid within the condenser, a tensional force acts outwards, of amount $(1/8\pi)\mathbf{E}^2\,\mathrm{grad}\,K$. The latter result is arrived at if, in our fundamental equation (13), p. 95, the term

$$\frac{1}{8\pi}\,\mathrm{grad}\left(\mathbf{E}^2\frac{dK}{d\sigma}\sigma\right)$$

in the force density, which is the term characteristic of electrostriction, is neglected. We can see that this procedure, while leading to the correct value for the total pressure difference $p' - p_0$, may give a distribution of pressure which in detail is completely altered. As a rule, in fact, according to (15*b*) the force which acts on the stratum of liquid between the plates A and B is not a tension at all, but a pressure; for E_n in our arrangement is nil. But, in addition to this, there is in the interior of the liquid at the lower end of the condenser (where the field $\mathbf{E}$ is to a high degree non-homogeneous) a marked pressure drop, given by (15*d*), which forces the liquid into the condenser, and over-compensates the pressure in the surface zone to such an extent that the result on the whole is given by (15*e*).

Finally, we consider the *force which acts on a charged metallic body within a liquid dielectric.* In this case, a peculiar point arises which makes the whole action of the electrostatic forces difficult to follow. The following example gives a simple illustration. Let a plate condenser be charged—at first in a vacuum—with the surface density $\pm\omega$. Then the first plate is subjected, on the side next the second, to an attractive force $\mathbf{E}_0^2/8\pi$ per square centimetre of its surface, where $4\pi\omega = |\mathbf{E}_0|$. The force arises from the intensity at the first plate due to the charge on the second plate. We now immerse the whole condenser in a liquid dielectric, keeping its charge unaltered. The attractive force therefore—as we know from the general energy considerations of § 2, p. 86—falls to $1/K$ of its old value. If we take the dielectric constant of the metal of the condenser plates as 1, it is at first sight out of the question that the intensity at the first plate should upon immersion fall to $1/K$ of its value, as we see by a glance at fig. 1, p. 71, showing the run of values of the vector $\mathbf{E}$ between the plates. In the liquid, no doubt, the intensity is lowered by this amount; but at the metal plates, where the charges are situated, its value has not altered at all. The force exerted on the charges on the first plate is therefore given as before by $\mathbf{E}_0^2/8\pi$. The diminution of the force acting on the plate is only brought about by the pressure action of the uncharged dielectric, which tends to force the condenser plates apart. In fact, according to (15*c*), the pressure

$$p' = \frac{1}{8\pi}\int_a^b \mathbf{E}^2 \frac{dK}{dx}\,dx + p - \frac{1}{8\pi}\mathbf{E}^2\frac{dK}{d\sigma}\sigma$$

acts on every square centimetre of plate surface. Of the three terms of this expression, the last two together, by (15*d*), have a constant value throughout the liquid, and in particular at the inner and outer surfaces of each plate, so that they give rise to no resultant force. The remaining integral is evaluated in (15′); and E_t is zero, while

E_n has the value $\mathbf{E}_0/K$, so that the pressure acting on the inner side of the plate has the value

$$\frac{1}{8\pi}(K-1)\frac{\mathbf{E}_0^2}{K}=\frac{1}{8\pi}\left(1-\frac{1}{K}\right)\mathbf{E}_0^2.$$

The whole force on the plate is found by subtracting this pressure from the Coulomb force $\mathbf{E}_0^2/8\pi$ acting on the charges. Not till this is done do we obtain the correct value of the total force, as required by the principle of energy, viz.

$$\frac{1}{8\pi}\cdot\frac{1}{K}\mathbf{E}_0^2.$$

8. The Maxwell Stresses.

The Faraday-Maxwell field theory has no dealings with forces acting at a distance, but regards all actions as transmitted continuously from one body to another through the electromagnetic field. The conception of the continuous transmission of force through a body is familiar enough—we may instance the example of a stretched elastic spring. Guided by this conception, Faraday sought the seat of electromagnetic action in a peculiar state of stress in the region of space occupied by electric or magnetic lines of force. This state of stress is the means by which force is conveyed from one charge to another. Starting from a system which is in electrostatic equilibrium, imagine it to be divided into two parts by an arbitrary closed surface S, and let one of the parts, say that enclosed by S, be denoted by 1, the remainder by 2. Then according to Faraday's idea the total force exerted by 2 on 1 must in some sense or other pass through this surface, and that quite irrespective of the fact that the surface may lie partly or even wholly in empty space. The idea was developed in a rigorous way by Maxwell, who showed that the total force exerted on the part 1 (the density $\mathbf{f}_e$ of this force being given by (13), p. 95), viz.

$$\mathbf{F}=\int_{(1)}\mathbf{f}_e dV,$$

can be represented by surface forces, acting on the boundary S of this part. (Note first that the expression for $\mathbf{F}$ just given actually contains only the force exerted by 2 on 1. For the mutual action between two charges, which both lie within 1, in consequence of the equality of action and reaction can contribute nothing to the resultant force.) We therefore denote by $\mathbf{T}dS$ a force, which is to act on the element of surface of the boundary of 1, and we assert with Maxwell that *it is possible so to transform the expression for* $\mathbf{F}$ *that we shall have*

$$\mathbf{F}=\iiint \mathbf{f}dV=\iint \mathbf{T}dS. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The action of 2 upon 1 is equivalent to the action of the surface forces $\mathbf{T}$. *From the point of view of the field theory, values of* $\mathbf{T}$ *to be allowable must depend only on the quantities defining the field at the place where the surface element is situated, and on the orientation of* dS (*the direction of its normal*) *with respect to the direction of the field.*

By Gauss's theorem, the transformation can be effected provided we can express the components of $\mathbf{f}$ in the form of a divergence. We shall carry out the process for the x-component of $\mathbf{F}$; i.e. we shall try to find quantities T_{xx}, T_{xy}, T_{xz} such that we have

$$f_x = \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z}. \quad . \quad . \quad . \quad . \quad (16a)$$

Corresponding equations will then hold for the y- and z-components of $\mathbf{f}$ also.

When we have succeeded in making the transformation, the components of the surface force $\mathbf{T}$ are given at once by Gauss's theorem:

$$\left.\begin{aligned} T_x &= T_{xx}\cos(n, x) + T_{xy}\cos(n, y) + T_{xz}\cos(n, z), \\ T_y &= T_{yx}\cos(n, x) + T_{yy}\cos(n, y) + T_{yz}\cos(n, z), \\ T_z &= T_{zx}\cos(n, x) + T_{zy}\cos(n, y) + T_{zz}\cos(n, z). \end{aligned}\right\} \quad (17)$$

To carry out the transformation indicated in (16) we put for brevity

$$\frac{1}{K}\,\frac{dK}{d\sigma}\,\sigma = \beta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17a)$$

$$\mathbf{f} = \rho\mathbf{E} - \frac{1}{8\pi}\mathbf{E}^2 \operatorname{grad} K + \frac{1}{8\pi}\operatorname{grad}(\beta K\mathbf{E}^2).$$

Further, we replace ρ by div $\mathbf{D}/4\pi$, and in the second term of $\mathbf{f}$ make use of the identity

$$\mathbf{E}^2 \operatorname{grad} K = \operatorname{grad}(\mathbf{E}^2K) - 2(K\mathbf{E}\operatorname{grad})\mathbf{E} - 2K[\mathbf{E}\operatorname{curl}\mathbf{E}]. \quad (17b)$$

Since $\mathbf{E}$ is irrotational, we find

$$f_x = \frac{1}{4\pi}E_x\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) - \frac{1}{8\pi}\,\frac{\partial}{\partial x}(\mathbf{ED})$$
$$+ \frac{1}{4\pi}\left(D_x\frac{\partial E_x}{\partial x} + D_y\frac{\partial E_x}{\partial y} + D_z\frac{\partial E_x}{\partial z}\right) + \frac{1}{8\pi}\,\frac{\partial}{\partial x}(\beta K\mathbf{E}^2),$$

or

$$8\pi f_x = \frac{\partial}{\partial x}(2E_xD_x - \mathbf{ED} + \beta K\mathbf{E}^2)$$
$$+ \frac{\partial}{\partial y}(2E_xD_y) + \frac{\partial}{\partial z}(2E_xD_z).$$

Thus f_x is expressed in the form required by (16a). By cyclical changes we can now obtain the components of Maxwell's stress tensor T:

$$\mathrm{T} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

$$\begin{bmatrix} \frac{K}{8\pi}(E_x^2-E_y^2-E_z^2+\beta \mathbf{E}^2) & \frac{K}{4\pi}E_xE_y & \frac{K}{4\pi}E_xE_z \\ \frac{K}{4\pi}E_yE_x & \frac{K}{8\pi}(E_y^2-E_z^2-E_x^2+\beta \mathbf{E}^2) & \frac{K}{4\pi}E_yE_z \\ \frac{K}{4\pi}E_zE_x & \frac{K}{4\pi}E_zE_y & \frac{K}{8\pi}(E_z^2-E_x^2-E_y^2+\beta \mathbf{E}^2) \end{bmatrix}. \qquad (18)$$

These quantities T_{xx}, T_{xy}, . . ., when inserted in (17), give the system of surface forces $\mathbf{T}$ which are required.

We shall now consider the connexion between the field vector $\mathbf{E}$ and the Maxwell stresses a little more closely. Their relation to one another turns out to be much simpler than might be expected from the appearance of the expressions in (18). In the first place, the terms containing β (the function which expresses the variability of the dielectric constant with position) occur only in the diagonal of T, and contribute a hydrostatic pressure of amount $K\beta\mathbf{E}^2/8\pi$, which acts at right angles to dS. This in fact could have been seen at once directly from the original form of $\mathbf{f}$. We shall now disregard this part, which in a vacuum is always nil, and discuss only that part of (18) which is left when we put $\beta = 0$. We fix our attention upon a definite element of surface, and choose the co-ordinate axes so that the positive axis of x is in the direction of the intensity $\mathbf{E}$, and the axis of z is at right angles both to the normal n to the surface element, and to the intensity. If further we denote by θ the angle between the normal and the intensity, and by E the absolute value of the intensity, then (fig. 3) we have

$$E_y = 0,\ E_z = 0,\ \cos(n, x) = \cos\theta,\ \cos(n, y) = \sin\theta,\ \cos(n, z) = 0.$$

Hence, by (17) and (18), the surface force $\mathbf{T}$ is given by

$$\left.\begin{aligned} T_x &= \frac{K}{8\pi}\mathbf{E}^2\cos\theta, \\ T_y &= -\frac{K}{8\pi}\mathbf{E}^2\sin\theta, \\ T_z &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

We thus have the following simple construction for the vector $\mathbf{T}$: *the absolute value of* $\mathbf{T}$ *is*

$$|\mathbf{T}| = \frac{K}{8\pi}\mathbf{E}^2,$$

at whatever angle the element of surface is inclined to the direction of

the field. We obtain the unit vector in the direction of **T** *by taking the image of the vector* **n** *in the direction of* **E** (fig. 3). In fact, the components of the unit vector so constructed, in the x, y, z directions, are $\cos\theta$, $-\sin\theta$, 0, as is required by (19). As need scarcely be said, the use of a special co-ordinate system in no way affects the generality of the result, it merely makes the proof simpler.

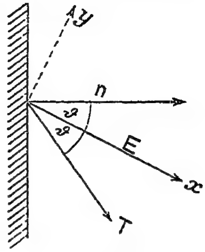

Fig. 3

According to this construction, the angle between the surface normal **n** and the force **T** is always bisected by the intensity **E**. By turning the surface element so that it makes various angles with the direction of the field, we therefore obtain the following results. If the field **E** is parallel to **n**, **T** also is in the direction of **n**, and we have pure tension. If we turn **n** away from **E**, **T** turns to the opposite side through an equal angle. For $\theta = 45°$, **T** lies in the plane of the surface element, so that the stress transmitted is pure shearing stress. On further increase of θ, the shearing stress again diminishes, in this case in favour of a pressure component, and the pressure alone remains when **n** is at right angles to **E**. Thus **T** is antiparallel to **n** with reference to the direction of **E**. During the whole of the above turning process. the magnitude of **T** remains constant. The algebraic sign of **E** does not affect **T**. *The stress tensor corresponds to pure tension when* **E** *is at right angles to* dS, *and to pure pressure when* **E** *is in the plane of* dS.

As another illustration of the circumstances, we shall consider briefly the force which two point charges exert on each other, for the cases when the two charges are equal (repulsion), or equal and opposite (attraction). We place the two charges on the axis of x at distances $+a$ and $-a$ from the origin. For the partial volume 1 (p. 104) of our system we take the hemisphere formed by the plane of yz and the left half of a very large sphere whose centre is at the origin. The surface force on the spherical part of the surface contributes nothing to the result, since on it $|\mathbf{T}|$ tends to zero like $1/R^4$. There remains only the action transmitted across the plane of symmetry.

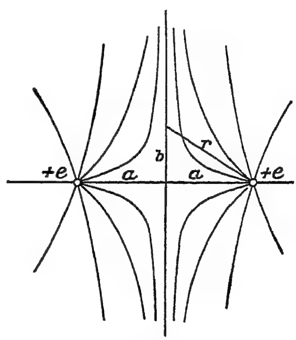

Fig. 4*a*

For *equal* charges ($e_1 = e_2 = e$), the field at any point on this plane (fig. 4*a*) is everywhere parallel to dS, so that the action on the plane is a pure pressure. We have in fact

$$E_x = 0,\ E_y = 2\frac{e}{Kr^2}\,\frac{y}{r},\ E_z = 2\frac{e}{Kr^2}\,\frac{z}{r}.$$

If then $b = \sqrt{y^2 + z^2}$ is the distance of a point on the plane of symmetry from the origin, we have on that plane

$$|\mathbf{T}| = \frac{K}{8\pi}\,\mathbf{E}^2 = \frac{1}{2\pi}\,\frac{e^2b^2}{Kr^6} = \frac{e^2}{2\pi K}\,\frac{b^2}{(a^2+b^2)^3}.$$

The area of the ring of breadth db is $\pi d(b^2)$. On putting $b^2 = \lambda$, we find for the total pressure on the plane

$$\begin{aligned}\mathbf{F} = \int \mathbf{T}dS &= \frac{e^2}{2K}\int_0^\infty \frac{\lambda d\lambda}{(a^2+\lambda)^3}\\ &= \frac{e^2}{2K}\int_0^\infty \left\{\frac{1}{(a^2+\lambda)^2} - \frac{a^2}{(a^2+\lambda)^3}\right\} d\lambda,\\ &= \frac{e^2}{2K}\left| -\frac{1}{a^2+\lambda} + \tfrac{1}{2}\,\frac{a^2}{(a^2+\lambda)^2}\right|_0^\infty,\end{aligned}$$

so that $\mathbf{F}$ is equal, as it should be, to the Coulomb repulsion

$$\frac{e^2}{K(2a)^2}.$$

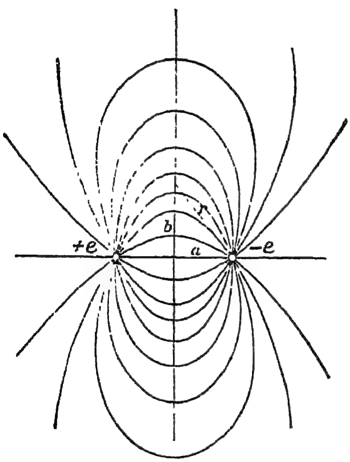

Fig. 4b

For *equal and opposite charges* ($e_1 = -e_2 = e$) the lines of force are everywhere perpendicular to the central plane (fig. 4b). The force is pure tension; and we have

$$E_x = 2\,\frac{e}{Kr^2}\,\frac{a}{r},\quad E_y = 0,\quad E_z = 0;$$

so that

$$|\mathbf{T}| = \frac{e^2}{2\pi K}\,\frac{a^2}{(a^2+b^2)^3},$$

and therefore $$|\mathbf{F}| = \frac{e^2}{2K}\int_0^\infty \frac{a^2 d\lambda}{(a^2+\lambda)^3} = \frac{e^2}{2K}\left| -\tfrac{1}{2}\,\frac{a^2}{(a^2+\lambda)^2}\right|_0^\infty = \frac{e^2}{K(2a)^2}.$$

From figs. 4a and 4b, which show how the lines of force run, we can recognize at once how the Maxwell stresses act: tension in the direction of the lines of force (fig. 4b), pressure in the perpendicular direction (fig. 4a).

CHAPTER VI

The Steady Electric Current

1. Ohm's Law. Joule's Law.

Among the equations which have been used in the preceding chapters to describe the electrostatic field, some will continue to be valid for more general electromagnetic processes. These are: the relation between the vectors $\mathbf{E}$ and $\mathbf{D}$; the equation connecting the divergence of $\mathbf{D}$ and the density of electric charge; and the expression for the density u of the electric energy; that is,

$$\mathbf{E} = K\mathbf{D}, \quad \operatorname{div} \mathbf{D} = 4\pi\rho, \quad u = \frac{1}{8\pi}(\mathbf{ED}).$$

Two relations belong exclusively to electrostatics, viz.

$$\mathbf{E} = -\operatorname{grad} \phi; \quad \phi = \text{const. in conductors};$$

which express the facts that $\mathbf{E}$ is irrotational, and that it vanishes within homogeneous conductors of electricity. We shall now give up the latter of these two conditions. To take an experimental example, suppose that we join the two coatings of a charged condenser, at different potentials ϕ_1 and ϕ_2, by a wire. As soon as the connexion has been made, the potential in the wire is certainly not constant, since it has the values ϕ_1 and ϕ_2 at its ends. Hence an electric field is set up within the wire, and the condition of electrostatic equilibrium is no longer satisfied. In point of fact the charges $+e$ and $-e$ tend to neutralize each other by the passage of electricity through the wire. Equilibrium is not established again until the charges of the condenser, and with them the field, have vanished. While the process of neutralization is going on, an electric current flows in the wire, of strength

$$i = -\frac{de}{dt}.$$

That there is actually something going on in the wire while the charge is changing can be inferred from the facts that heat is developed in the wire, and that a magnetic field exists in its neighbourhood The occurrence of the magnetic field is a complication in the process

the full consideration of which we defer till later chapters. So long as the strength i of the current remains constant, the magnetic field which it generates does not change. Hence, up to a certain point, we can deal with the laws of *steady* currents (i.e. those which do not vary with the time), without taking account of the accompanying magnetic field. In our example, as it happens, we can only *approximate* to the condition of steadiness, viz. by taking the resistance of the wire and the capacity of the condenser as great as possible. By purely electrostatic means, it is not possible to realize a *perfectly* steady current. This can only be done by the use of appliances foreign to pure electrostatics (e.g. voltaic cells, accumulators, or thermo-elements). We shall have to return to these in the sections which follow.

Meantime we continue the consideration of the almost steady case of a condenser of very great capacity. By measuring the rate of discharge ($-de/dt$) we can establish *Ohm's law*,

$$i = \frac{\phi_2 - \phi_1}{R}. \qquad (1)$$

R is called the resistance of the wire, and depends only on its *dimensions* and the nature of the *material* of which it is made. If l is the length of the wire, and S its cross-section, then

$$R = \frac{1}{\sigma}\frac{l}{S}, \qquad (1a)$$

where the quantity σ depends only on the *material* of the wire; σ is called the *specific conductivity*. Its reciprocal $1/\sigma$ is called the *specific resistance* (per unit volume), and is equal to the resistance of a cube of side 1 cm. ($l = 1$, $S = 1$). Equation (1) expresses Ohm's law in its immediately observable form. This form cannot be used in a field theory, since no results except those which refer only to the immediate neighbourhood of a single point can be used as the primary propositions of such a theory. In order to discover the differential form of Ohm's law, which this condition requires, we make the assumption that the same relation as has been found in the first instance for the wire as a whole holds also for any arbitrary element of its volume. As such an element we choose a small cylinder, having its length dl in the direction of the field, and its cross-section dS perpendicular to the field. Then, according to (1), if $d\phi$ is the potential difference between the ends of dl, we have

$$i = -\frac{d\phi}{R}, \quad \frac{1}{R} = \sigma\frac{dS}{dl},$$

so that

$$\frac{i}{dS} = -\sigma\frac{d\phi}{dl}.$$

We write this equation vectorially in the form

$$\mathbf{i} = \sigma\mathbf{E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the vector $\mathbf{i}$ thus introduced being called the *current density*. The current density $\mathbf{i}$ is therefore so defined that $i_n dS$ is the quantity of electricity which passes through the surface element dS in the direction of its normal $\mathbf{n}$ in the unit of time. Equation (2) is the required differential form of Ohm's law; it also gives a definition of σ suitable for the field theory. In this form of the law there will be nothing to alter, even when we come to consider processes which vary with the time, while (1) applies to steady currents only.

It must be observed that (2) only holds for isotropic substances, that is, for substances whose properties do not depend on direction. In anisotropic bodies (e.g. crystals, or materials in a state of strain) the conductivity σ depends in general on the direction of the current. In such cases σ is not a scalar, but a symmetric tensor (p. 47). Equation (2) is then to be read as a tensor equation, in which the vectors $\mathbf{i}$ and $\mathbf{E}$ are no longer parallel, but are connected by three equations giving i_x, i_y, i_z, of the type

$$i_x = \sigma_{xx}E_x + \sigma_{xy}E_y + \sigma_{xz}E_z.$$

In what follows, however, we shall confine ourselves to isotropic substances.

Joule's law defines the quantity of heat developed in a wire traversed by a current. In our case of a condenser short-circuited by a wire the heat developed per unit time is

$$Q = -(\phi_2 - \phi_1)\frac{de}{dt} = (\phi_2 - \phi_1)i = Ri^2.$$

Thus the field energy which disappears during the discharge of the condenser reappears in equivalent quantity as "Joule heat", Ri^2. Here, let it be emphasized again, we are neglecting the change in the magnetic field energy which is involved in the change of the current strength. By taking that part of the energy into account, we shall be led later (p. 139) to Faraday's law of induction. The last equation also is translated at once into differential form, by applying it to the volume element $dV = dl \, . \, dS$ which we used above. We thus get

$$R = \frac{1}{\sigma}\frac{dl}{dS}, \quad \text{and} \quad i = \mathbf{i}dS = \sigma\mathbf{E}dS.$$

The Joule heat per unit volume therefore becomes

$$\frac{Q}{dV} = \sigma\mathbf{E}^2 = (\mathbf{iE}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

so that (iE) is the heat developed per unit volume and unit time. This result, as also the integral law $Q = (\phi_2 - \phi_1)i$, can be deduced from the principle of energy. The general proof will be given in a later section (p. 145). In the case, however, when the current strength changes only very slowly, we can still regard the field of **E** as approximately irrotational, and for the rate of change of the field energy, i.e. of

$$U = \frac{1}{8\pi}\int (\mathbf{ED})dV,$$

take the value used in electrostatics (equation (10), p. 93),

$$\frac{dU}{dt} = \int \phi \frac{\partial \rho}{\partial t} dV.$$

In the present case ρ will differ from zero even in the interior of the conductor. We consider only that part of the variation of ρ, the charge density, which is due to a conduction current **i**. From the definition of **i**, we have

$$\frac{d}{dt}\int \rho\, dV = -\int i_n dS,$$

and therefore, by Gauss's theorem,

$$\frac{\partial \rho}{\partial t} = -\text{div}\,\mathbf{i}.$$

But we have the general equation

$$-\phi\,\text{div}\,\mathbf{i} = -\text{div}\,(\phi\mathbf{i}) + (\mathbf{i}, \text{grad}\,\phi).$$

Thus, by integration over the whole system, we find

$$\frac{dU}{dt} = \int (\mathbf{i}, \text{grad}\,\phi)\, dV = -\int (\mathbf{iE})\, dV.$$

This is exactly the result to be expected from (3), viz.: the decrease in the field energy per second is equal to the Joule heat developed in the whole system. It should be particularly noted, however, that the proof is not rigorous, since in fields which vary with the time **E** ceases to be irrotational.

2. Conduction Current. Displacement Current. Polarization Current.

For a steady current, it is a general principle that the same current **i** flows through every cross-section of the conducting circuit. An analogous but more general result is that, in a volume distribution of steady current in space, the current density **i** is everywhere solenoidal. For a source of **i** at any point would imply a time rate of

change in the charge density, and consequently in the field $\mathbf{E}$. For steady currents, therefore, we have the conditions: (*a*) $\operatorname{div} \mathbf{i} = 0$ within any conductor; (*b*) i_n is continuous at the boundary between two conductors.

At the surface forming the common boundary of two conductors we have accordingly the two boundary conditions: continuity of the normal component of $\sigma\mathbf{E}$, and continuity of the tangential component of $\mathbf{E}$. Hence, in steady flow of electricity, the lines of flow at the common boundary of two conductors of conductivities σ_1 and σ_2 are refracted in the same way as the lines of displacement at the boundary between two insulators of dielectric constants K_1 and K_2, provided $K_1 : K_2$ is equal to $\sigma_1 : \sigma_2$ (cf. fig. 2, p. 76).

A variable or non-steady current, such as the current in our first example of discharge of a condenser, is in general associated with a time rate of change of the density of charge. In fact, by Gauss's theorem we have

$$\operatorname{div} \mathbf{i} = -\frac{\partial \rho}{\partial t}.$$

But since on the other hand we have always

$$4\pi\rho = \operatorname{div} \mathbf{D},$$

we obtain for a current varying with the time

$$\operatorname{div} \mathbf{i} = -\frac{1}{4\pi} \operatorname{div} \frac{\partial \mathbf{D}}{\partial t}. \quad \ldots \ldots \quad (4)$$

If then we introduce a vector $\mathbf{c}$, with the definition

$$\mathbf{c} = \mathbf{i} + \frac{1}{4\pi} \frac{\partial \mathbf{D}}{\partial t}, \quad \ldots \ldots \quad (5)$$

we shall always have $\operatorname{div} \mathbf{c} = 0$, within a conductor; and c_n continuous at its boundary.

By the addition of the *displacement current* $\frac{1}{4\pi} \frac{\partial \mathbf{D}}{\partial t}$, the conduction current $\mathbf{i}$ becomes the solenoidal *total current* $\mathbf{c}$. The introduction of this solenoidal total current is due to Maxwell. Later, in general electrodynamics, we shall find it to be of fundamental importance. A simple illustration of the idea may again be drawn from the example of a plate condenser short-circuited by a wire. The conduction current flowing in the wire, i.e.

$$i = \frac{de}{dt},$$

terminates at the coatings of the condenser. But if S is the area

of either plate, and ω the surface density of its charge, the displacement in the insulator is

$$\mathbf{D} = 4\pi\omega = 4\pi \frac{e}{S}.$$

Hence, so long as the current i flows, $\mathbf{D}$ changes in such a way that the total displacement current in the insulator is

$$S \cdot \frac{1}{4\pi} \frac{\partial \mathbf{D}}{\partial t} = \frac{\partial e}{\partial t} = i.$$

It is therefore equal to the conduction current in the wire. By means of the displacement current the conduction current is continued under solenoidal conditions into the adjoining insulator.

To give a general interpretation of (5), we must attribute to every body both a conductivity σ and a dielectric constant K. We then have

$$\mathbf{c} = \sigma \mathbf{E} + \frac{K}{4\pi} \frac{\partial \mathbf{E}}{\partial t}.$$

The work done on this total current by the field $\mathbf{E}$, per unit volume and unit time, thus becomes

$$(\mathbf{c}\mathbf{E}) = \sigma \mathbf{E}^2 + \frac{\partial}{\partial t}\left(\frac{K}{8\pi}\mathbf{E}^2\right).$$

The work appears on the right-hand side partly as Joule heat $\sigma\mathbf{E}^2$, partly as augmentation of the field energy $K\mathbf{E}^2/8\pi$.

In practical applications the two terms of which $\mathbf{c}$ is composed are always of different orders of magnitude. In perfect insulators the displacement current alone is present. In all metallic conductors, on the contrary, the conduction current is so large compared with the displacement current, that we can practically always neglect the latter. Only with very rapidly varying fields (waves of visible light, or shorter waves) does the displacement current become sensible in metals.

The two equations

$$\rho = \frac{1}{4\pi} \operatorname{div}(K\mathbf{E})$$

and

$$\frac{\partial \rho}{\partial t} = - \operatorname{div}(\sigma \mathbf{E})$$

lead to an important deduction with respect to the time rate of dispersal of a charge originally present in the interior of a conductor. If we regard σ and K as constant, and eliminate $\mathbf{E}$, the equations give

$$\frac{\partial \rho}{\partial t} = - \frac{4\pi\sigma}{K} \rho,$$

from which, by integration, we find

$$\rho = \rho_0 e^{-t/\theta}, \quad \theta = \frac{K}{4\pi\sigma}.$$

The time θ, called the "modulus of decay"*, is the time it takes the charge density to diminish to $1/e$ of its original value. It also indicates the order of magnitude of the time required to establish electrostatic equilibrium. We give a few numerical examples, but it should be noted that there is some uncertainty with respect to the value of K in metals. We know of nothing, however, which would lead us to assign to metals a dielectric constant of an order of magnitude differing from 1. We therefore get the order of magnitude of the modulus of decay also, if we are content with the value of θ/K, i.e. $1/4\pi\sigma$, as given in the following Table.

	R_0	$\sigma = 9 \times 10^{11}/R_0$	$1/4\pi\sigma = \theta/K$
Copper ..	$1{\cdot}7 \times 10^{-6}$ ohm	53×10^{16} sec.$^{-1}$	$0{\cdot}15 \times 10^{-18}$ sec.
Platinum ..	$10{\cdot}7 \times 10^{-6}$,,	$8{\cdot}4 \times 10^{16}$,,	$0{\cdot}95 \times 10^{-18}$,,
Bismuth ..	120×10^{-6} ,,	$0{\cdot}75 \times 10^{16}$,,	$10{\cdot}6 \times 10^{-18}$,,

The values of the modulus of decay are therefore excessively small. This is as much as we can infer from such a result as $\theta = 10^{-18}$ sec. In fact, for processes of such rapidity the methods of the present volume become meaningless. On the other hand, the numerical values of σ deserve attention. They are of dimensions sec.$^{-1}$, so that σ is a frequency. In all formulæ describing the behaviour of metals in alternating fields, the ratio of the frequency ν of the field to the frequency given by σ is the critical number. It may be noted that to reach a frequency as high as 10^{15} sec.$^{-1}$ we must go as far as ultra-violet light of wave-length $0{\cdot}3\mu$.

The displacement current $\frac{1}{4\pi}\frac{\partial \mathbf{D}}{\partial t}$ is really made up of two components of quite different character: thus

$$\frac{1}{4\pi}\frac{\partial \mathbf{D}}{\partial t} = \frac{1}{4\pi}\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t},$$

i.e. the total displacement current is the sum of the displacement current *in vacuo*, and the polarization current. According to our definition (p. 72) of the vector $\mathbf{P}$, it seems perfectly natural that the time rate of variation of $\mathbf{P}$ should occur in an equation as a current density. In fact, if $\mathbf{P}$ changes by the amount $d\mathbf{P}$, this simply means by the definition that a quantity of electricity $(d\mathbf{P} \cdot \mathbf{n})dS$ has crossed the element of area dS.

* Ger. *Relaxationszeit.*

The new part, and the part which is characteristic of the Maxwellian theory, is accordingly the "*displacement current in vacuo*",

$$\frac{1}{4\pi}\frac{\partial \mathbf{E}}{\partial t}.$$

The splitting up of $\mathbf{D}$ into $\mathbf{E}$ and $4\pi\mathbf{P}$, which seems natural to us at the present day, is utterly alien to the original Faraday-Maxwell conception. There was at one time a tendency towards the view that even in empty space, exactly as in a dielectric, positive and negative charges are bound to one another quasi-elastically, the vector $\mathbf{E}/4\pi$ being then regarded as a polarization of empty space. The discussion of this idea, and attempts to construct an æther model to fit it, have played a great part in the history of electrodynamics. These efforts, however, have not proved fruitful. Only the polarization current can be interpreted as an actual transport of electric charge. If then we write the solenoidal total current in the form

$$\mathbf{c} = \mathbf{i} + \frac{\partial \mathbf{P}}{\partial t} + \frac{1}{4\pi}\frac{\partial \mathbf{E}}{\partial t},$$

the first two terms represent an actual transport of charge, which by the addition of the third term is made up to a solenoidal vector.

3. Impressed Forces and Electromotive Force.

When applying Ohm's law in the form given above, p. 111,

$$\mathbf{i} = \sigma\mathbf{E},$$

we have always restricted ourselves to homogeneous conductors. The law ceases to hold in non-homogeneous conductors, and in the transitional layer between one conductor and another. According to the above equation we should always have $\mathbf{i} = 0$ when $\mathbf{E} = 0$. In point of fact observation shows that in general, at a place where there is non-homogeneity, the intensity must not vanish if equilibrium is to be maintained, i.e. if there is to be no current. It follows that at such a place the intensity $\mathbf{E}$ is not the only cause of the occurrence of a current. There must also be other forces present, which tend to send a current through the conductor. We shall take account of these forces by introducing a vector $\mathbf{E}^{(e)}$, and extending Ohm's law to the form

$$\mathbf{i} = \sigma(\mathbf{E} + \mathbf{E}^{(e)}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For brevity, we call $\mathbf{E}^{(e)}$ the "*impressed* force", or "*applied* force", (though this use of the word force is not quite correct); $\mathbf{E}^{(e)}$ is therefore a vector which depends on the kind of non-homogeneity which

exists at the place in question, and which combines with the intensity $\mathbf{E}$ to give rise to the current density $\mathbf{i}$ at the place in accordance with equation (6).

Although the relation stated in (6) is sufficient for the purposes of a formal theory, the field of the impressed forces $\mathbf{E}^{(e)}$ being simply regarded as given, it will help to engender a more vivid apprehension of the subject if we consider briefly how that field arises in some special cases.

The example of an impressed force in which the actions occurring can be followed most easily is that in which the conductor is a dilute solution of a strong electrolyte (e.g. HCl), and the non-homogeneity a variation in its concentration from point to point. To begin with, let there be no electric field present. Then a process of diffusion will set in, which tends to smooth out the differences of concentration. Now the electrolyte is dissociated, practically completely so, into H^+ and Cl^- ions, which diffuse independently of one another. The mobility, and therefore also the rate of diffusion, is, however, much greater for the H^+ ions than for the Cl^- ions. The effect of this is to produce an electric current in the direction of diminishing concentration, since more H^+ ions than Cl^- ions are set in motion towards the places of weak concentration. We can recognize in this case that an impressed force $\mathbf{E}^{(e)}$ can be caused by diffusion. Further, the diffusion current causes the dilute parts of the solution to become positively charged, and the concentrated parts negatively, and so gives rise to an electric field in such a direction that the diffusion of the H^+ particles is checked and that of the Cl^- ions accelerated. Finally, a state of electrical equilibrium will be reached, in which the difference in the rates of diffusion of the two kinds of ions is exactly compensated by this field. We then have a state of affairs in which there is no current, characterized by an electric field $\mathbf{E}$, which exactly compensates the impressed field $\mathbf{E}^{(e)}$: $\mathbf{E} + \mathbf{E}^{(e)} = 0$.

Between the vectors $\mathbf{E}$ and $\mathbf{E}^{(e)}$ there exists a fundamental difference, which must be explicitly emphasized. The impressed field $\mathbf{E}^{(e)}$ is present only in the interior of the electrolyte, where there is a finite gradient of concentration; in the surrounding vacuum or dielectric $\mathbf{E}^{(e)}$ is always zero. The behaviour of the electrostatic field $\mathbf{E}$ is quite different. Of course, in the interior of the electrolyte, when there is no current, we have everywhere $\mathbf{E} = -\mathbf{E}^{(e)}$. But outside of it, $\mathbf{E}$ is determined by the principles of electrostatics (continuity of the tangential components of $\mathbf{E}$ at the bounding surface). Hence $\mathbf{E}$ is in all circumstances irrotational; on the contrary the integral $\oint(\mathbf{E}^{(e)} d\mathbf{s})$ taken round a closed curve is only zero in special exceptional cases, viz. when (1) the impressed forces are so distributed that they can be compensated within the conductor by an electrostatic field,

and (2), the path of integration lies wholly within this conductor. Only in such cases can we speak of an "impressed (or applied) potential difference".

We shall now obtain a formal expression for the impressed force produced in an electrolyte by a concentration gradient.

Let n be the number of HCl molecules per c.c., and therefore also the number of H^+ ions and the number of Cl^- ions, n being a given function of position; and let D_+ and D_- be the diffusion constants, B_+ and B_- the mobilities, of the two kinds of ions, and e, $-e$ their charges.

The definitions of D_+ and B_+ are as follows: $(-)D_+ \operatorname{grad} n$ is the current vector due to diffusion, for the $+$ ions (the number of particles crossing 1 sq. cm. per sec.); $B_+\mathbf{F}$ is the velocity acquired by an ion under the action of a force $\mathbf{F}$; and similarly for D_- and B_-.

The total current density under the action of diffusion and a field $\mathbf{E}$ is accordingly

$$\mathbf{i} = -e(D_+ - D_-) \operatorname{grad} n + e^2(B_+ + B_-)n\mathbf{E},$$

or

$$\mathbf{i} = e^2 n(B_+ + B_-) \left\{\mathbf{E} - \frac{1}{e}\frac{D_+ - D_-}{B_+ + B_-} \operatorname{grad} (\log n)\right\}.$$

We thus arrive at Ohm's law, exactly in the form of (6), p. 116. Moreover, as comparison with that equation shows, we have for our binary electrolyte,

$$\sigma = e^2 n(B_+ + B_-)$$

and

$$e\mathbf{E}^{(e)} = -\frac{D_+ - D_-}{B_+ + B_-} \operatorname{grad} (\log n).$$

In our example, therefore, $\mathbf{E}^{(e)}$ points in the direction in which the concentration falls fastest. Between any two points where the concentrations are n_1 and n_2, we have an *impressed potential difference* $E_1^{(e)} - E_2^{(e)}$, for which, by integration along a curve lying entirely in the solution, we find

$$e(E_2^{(e)} - E_1^{(e)}) = +e\int_2^1 E_s^{(e)} ds = \frac{D_+ - D_-}{B_+ + B_-} \log \frac{n_2}{n_1}.$$

In the state when there is no current, an electrostatic potential difference is present, which is equal and opposite to the impressed one:

$$\phi_2 - \phi_1 = -(E_2^{(e)} - E_1^{(e)}).$$

For the sake of completeness, we may remark that by a very general theorem of statistical mechanics a relation exists between D and B of the form

$$D = kTB$$

($k =$ Boltzmann's constant $1{\cdot}37 \times 10^{-16}$, $T =$ absolute temperature); and that in electrochemistry the number

$$\nu = \frac{B_+}{B_+ + B_-}$$

is called the transport number. We therefore have

$$e(E_2^{(e)} - E_1^{(e)}) = kT(2\nu - 1) \log \frac{n_2}{n_1}.$$

The order of magnitude of this pressure difference may be seen from the numerical example: $\log (n_2/n_1) = 1$, $n_2 : n_1 = 2{\cdot}72$, $\nu = 1$, $T = 300$, $e = 4{\cdot}77 \times 10^{-10}$. The potential difference here is

$$\frac{1{\cdot}37 \times 10^{-16} \times 300}{4{\cdot}77 \times 10^{-10}} = 0{\cdot}86 \times 10^{-4} \text{ c.g.s. units} = 0{\cdot}026 \text{ volt.}$$

Another example of an impressed force presents itself when a metal and electrolyte are in contact with each other. For example, when a copper rod is dipped into a dilute solution of copper sulphate, a small quantity of copper first goes into solution in the form of Cu^{++} ions. An electric current therefore flows from the copper into the electrolyte. In this case the solution pressure of the copper causes an " impressed force ". The current does not continue, for it produces a negative charge on the copper, and a positive one in the solution, and these give rise to an electric field, directed towards the copper, in the layer of liquid immediately adjacent to it. In equilibrium, this field exactly compensates the " impressed force " of the solution pressure.

Here again we have, in equilibrium, as a consequence of the impressed potential difference, an equal electrostatic potential difference opposed to it,

$$\phi_{12} = \int_2^1 \mathbf{E}\, ds = \int_1^2 \mathbf{E}^{(e)}\, ds,$$

between the interior of the metal (1) and the homogeneous solution (2). The transition zone in the electrolyte, in which $E^{(e)}$ is sensible, is as a rule so narrow that we may almost speak of a potential " jump " ϕ_{12} at the boundary between metal and electrolyte. It is on this that the action of the voltaic cell effectively depends.

As a third instance of an impressed potential difference we may mention the contact of two different metals. In this case the dissimilarity between the motion of the electrons in the two metals produces a current, which continues until a definite potential difference has been set up between the metals. A more detailed discussion of this process is reserved for the sections on the electron theory of metals.

4. The Voltaic Circuit.

We consider next a so-called *circuit*, i.e. a number of different conductors connected to one another in series, which may contain, in themselves and at their ends, arbitrary impressed electric forces (intensities or field strengths). We shall assume, however, that the beginning (1) and the end (2) of the circuit (fig. 5) consist of the same material. In that case, when there is no current there exists between these ends a potential difference, which is given by the line integral of all the impressed intensities,

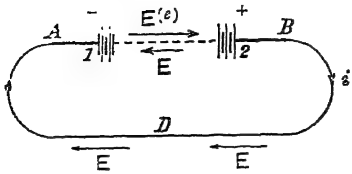

Fig. 5

$$\phi_{12} \equiv \phi_2 - \phi_1 = \int_1^2 E_s^{(e)} ds,$$

where the path of integration must lie entirely within the circuit.

If now the two ends of the circuit are brought into contact (by means of the wire BDA, which closes the circuit), electrostatic equilibrium at once becomes impossible. For the line integral $\oint \mathbf{E}^{(e)} ds$ is now different from zero, since $\mathbf{E}^{(e)} = 0$ at all points of the connecting wire BDA, so that

$$\oint \mathbf{E}^{(e)} ds = \int_1^2 \mathbf{E}^{(e)} ds = E_{12}^{(e)}.$$

It is therefore impossible to counterbalance $\mathbf{E}^{(e)}$ by an electrostatic field. Hence, by (6), p. 116, an electric current must begin to flow. When the current

$$\mathbf{i} = \sigma(\mathbf{E} + \mathbf{E}^{(e)})$$

has developed, we have steady conditions once more.

The characteristic property of a steady current is that it is solenoidal (div $\mathbf{i} = 0$); as for the field $\mathbf{E}$, we must have everywhere curl $\mathbf{E} = 0$. These two conditions are sufficient, when σ and $\mathbf{E}^{(e)}$ are given, to allow both $\mathbf{i}$ and $\mathbf{E}$ to be calculated.

We shall go through the necessary calculations for the case of a linear circuit. If S is the cross-section of the conducting circuit (perpendicular to the lines of flow) and ds an element of length in the direction of the path of the current, then

$$(\mathbf{i}\,d\mathbf{s}) = \frac{1}{S} i\,ds.$$

The solenoidal character of $\mathbf{i}$ is expressed here by the fact that the same current i must flow through every cross-section. If then, we

integrate equation (6), p. 116, over the whole ring-shaped closed circuit, from (2) along the wire BDA to (1), and then through the remaining part of the circuit back to (2), we obtain

$$i \oint \frac{ds}{\sigma S} = + \oint (\mathbf{E} + \mathbf{E}^{(e)})\, ds$$

$$= \oint \mathbf{E}^{(e)}\, ds = E_{12}^{(e)},$$

since the line integral of $\mathbf{E}$ must vanish. We write

$$\oint \frac{ds}{\sigma S} = R,$$

and call R the *resistance* of the whole conducting circuit. We thus have the result: *the product of the current strength and the resistance of the whole ring-shaped closed circuit is equal to the line integral of the impressed electric force.* On account of this property the quantity $E_{12}^{(e)}$ is also spoken of as the *electromotive force* (E.M.F.) of the closed circuit. In an open circuit it is not identical with the line integral unless the first and last component parts of the circuit are made of the same material.

When i has been determined, we can find $\mathbf{i}$ also, and accordingly $\mathbf{E}$, at every point of our linear circuit:

$$\mathbf{E} = \frac{\mathbf{i}}{\sigma} - \mathbf{E}^{(e)}.$$

In another limiting case, that of a conducting system extending to infinity, with $\mathbf{E}^{(e)}$ and σ any given continuous functions of position, the steady distribution of current is uniquely determined by the two conditions

$$\operatorname{div} \mathbf{i} = 0, \quad \operatorname{curl}(\mathbf{i}/\sigma) = \operatorname{curl} \mathbf{E}^{(e)};$$

while the field $\mathbf{E}$ is given uniquely by the equations

$$\operatorname{curl} \mathbf{E} = 0, \quad \operatorname{div}(\sigma \mathbf{E}) = -\operatorname{div}(\sigma \mathbf{E}^{(e)}).$$

We have still to consider the development of heat in the case of a steady current maintained by impressed forces:

$$Q = \int (\mathbf{i}^2/\sigma)\, dV = \int \mathbf{i}\, (\mathbf{E} + \mathbf{E}^{(e)})\, dV,$$

where the integral is to be taken over the whole region traversed by currents. Since $\mathbf{E} = -\operatorname{grad} \phi$, and

$$(\mathbf{i} \operatorname{grad} \phi) = \operatorname{div}(\mathbf{i}\phi) - \phi \operatorname{div} \mathbf{i},$$

the above equation, when the current is steady ($\operatorname{div} \mathbf{i} = 0$), gives

in the first place, by integration over the whole region containing currents,

$$Q = \int (\mathbf{i}\mathbf{E}^{(e)})\,dV - \int \phi i_n\,dS.$$

If the integration is carried out for the complete system, then $\mathbf{i} = 0$ at every point of the boundary of the integration space, so that we obtain

$$Q = \int (\mathbf{i}\mathbf{E}^{(e)})\,dV.$$

The total heat developed in a circuit is therefore equal to the work done by the impressed forces. Where the equivalent energy comes from depends entirely on the process which is responsible for the occurrence of $\mathbf{E}^{(e)}$. In the concentration cell the energy is obtained at the cost of the free energy which the concentrated solution possesses in greater degree than the dilute solution. In the voltaic cell, the energy comes from the chemical reactions connected with solution or separation; in the thermocouple, from the sources of heat which maintain the temperature difference of the junctions. In every case the work done, $+\int(\mathbf{i}\mathbf{E}^{(e)})\,dV$, is derived from sources of energy which lie outside the proper domain of electrostatics, just as the Joule heat lies outside that domain. We have here, therefore, the remarkable phenomenon of an electric field which is steady, i.e. does not vary with the time, but yet is the agency by which one kind of energy of a non-electrical nature is continuously converted into another, namely into heat.

Part III

THE ELECTROMAGNETIC FIELD

CHAPTER VII

Magnetic Vectors

1. Magnetic Intensities in Vacuo.

Prior to the discoveries of Oersted (1820) and Faraday (1831), magnetism as a branch of physical science was completely independent of the theory of electricity. It dealt with the mutual actions of permanent magnets, including the earth with its magnetic field. Nevertheless there is a far-reaching formal analogy between the magnetostatic and the electrostatic field, which in certain cases makes it possible to deal with them mathematically in a similar way. Electrostatics starts from Coulomb's law of force for charged proof bodies, and from the definition and realization of units of charge and intensity which this law renders possible. But this is just one of the points where the analogy between the electrostatic and the magnetostatic field fails, since a little proof sphere carrying a magnetic charge cannot be obtained. There are no "magnetically charged" bodies. We can certainly magnetize a piece of iron, i.e. cause it to be magnetically polarized, but we can never charge it with magnetism. There is a second essential difference, however, which makes up for this one when we have to investigate magnetic fields experimentally. While the electric polarization **P** of a dielectric can only be maintained by an external electric field and is proportional to the field **E**, there are substances with "permanent" magnetic polarization **I**, where **I** denotes (as will be seen later) the "magnetic moment per unit volume" or "intensity of magnetization". Certain substances, described as being magnetically hard, can be magnetized in such a way that, when the external field is not too strong, **I** only depends to a slight extent on **H**. In the preliminary explanations of this section we shall consider the limiting case of an ideal hard bar magnet, i.e. we assume its magnetization to have a given fixed value, independent of the external field.

With the help of small permanent bar magnets we can explore a magnetic field just as well as we can explore an electric field by means of a proof sphere. In the choice of the requisite units we also follow the guidance of the formal analogy with the electrostatic field. In exactly the same way as the charged test body in the electrical case, the magnetic needle in the present case acts both as an indicator and as a source (really a double source) of a magnetic field.

We shall begin by considering the method by which Gauss succeeded in measuring for the first time both the magnetization of such a bar magnet and the intensity of the earth's field, in absolute units. If we denote the magnetic moment of our small magnet by

$$\mathbf{M} = \int \mathbf{I} dV,$$

then in a homogeneous magnetic field $\mathbf{H}_0$ (say the earth's magnetic field) a couple $\mathbf{N}$ will act upon it, where

$$\mathbf{N} = [\mathbf{M}\mathbf{H}_0];\ N = |\mathbf{M}| . |\mathbf{H}_0| \sin\theta,$$

θ being the angle between $\mathbf{M}$ and $\mathbf{H}_0$.

If in particular the magnet is free to turn about an axis, and if Θ is its moment of inertia, and H_0 the component of the homogeneous external field perpendicular to the axis of rotation, then we have the equation of motion

$$\Theta \frac{d^2\theta}{dt^2} = -MH_0 \sin\theta.$$

For small oscillations ($\sin\theta \approx \theta$), by putting $\theta = C \sin 2\pi\nu t$ we find that the frequency is

$$\nu = \frac{1}{2\pi}\sqrt{\left(\frac{MH_0}{\Theta}\right)}.$$

This experiment therefore enables us to determine the *product* MH_0 in absolute measure.

Let us now consider the magnet itself as the source of a magnetic field $\mathbf{H}$. This field can be derived from a potential ϕ_m which (p. 23) is given (fig. 1) by

$$\phi_m = \frac{M}{r^2}\cos\psi.$$

Hence

$$H_r = -\frac{\partial\phi_m}{\partial r} = \frac{2M}{r^3}\cos\psi,$$

$$H_\psi = -\frac{1}{r}\frac{\partial\phi_m}{\partial\psi} = \frac{M}{r^3}\sin\psi.$$

We now (fig. 1) fix our bar magnet so that its length is at right angles to the homogeneous field $\mathbf{H}_0$ (of numerical value H_0), and place a small magnetic needle, which can rotate freely, at a point distant r_0 from the centre of the bar magnet in the direction of the moment $\mathbf{M}$ ($\psi = 0$). This needle will set itself in the direction of the resultant of $\mathbf{H}$ and $\mathbf{H}_0$ at that point, and will therefore be deflected from the direction of $\mathbf{H}_0$ through the angle α, where

$$\tan\alpha = \frac{|\mathbf{H}|}{|\mathbf{H}_0|} = \frac{2M}{r_0{}^3 H_0}.$$

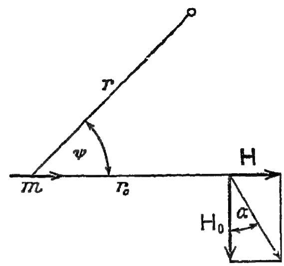

Fig. 1

By measuring α and r_0, we can therefore this time determine the *quotient* M/H_0 in absolute measure. By combining the two results, we obtain in absolute units both the earth's field H_0 and the moment M of our bar magnet. By means of the magnet thus standardized, we can now determine the magnitude and direction of a given magnetic field at any point, provided only the magnet itself is taken so small that the field in its immediate neighbourhood can be regarded as homogeneous. The source of the magnetic field $\mathbf{H}$ in this case is the magnetization $\mathbf{I}$ of our permanent magnet; and we have

$$\operatorname{div} \mathbf{H} = -4\pi \operatorname{div} \mathbf{I},$$

in exact agreement with the equation (p. 74) for the dielectric polarization

$$\operatorname{div} \mathbf{E} = -4\pi \operatorname{div} \mathbf{P}.$$

Further, in pure magnetostatics (i.e. in the absence of electric currents and with fields which do not vary with the time), we have $\mathbf{H} = -\operatorname{grad} \phi_m$. Thus $\mathbf{H}$ is irrotational in these circumstances. Its sources are situated at those places where the magnetization varies with position in such a way that its divergence is not zero; in a homogeneously magnetized bar magnet these places are at its ends, where the normal component of $\mathbf{I}$ gives rise to the surface divergence

$$(H_n)_{\text{magnet}} - (H_n)_{\text{vacuum}} = -4\pi I_n.$$

2. The Magnetic Field of Steady Currents.

According to Oersted's discovery an electric current is always accompanied by a magnetic field. The magnetic field of a straight wire of infinite length in which a current flows consists of lines of force in the form of circles surrounding the wire, with their planes perpendicular to it. The direction of $\mathbf{H}$ in one of these circles has the

right-handed screw relation to the direction of the current. The field in this case is not irrotational, the line integral

$$\oint H_s ds$$

not being zero for a path surrounding the wire. The survey of this field has shown that in fact the value of the line integral is directly proportional to the current threading the path of integration. If we write the factor of proportionality in the form $4\pi/c$, then all physical results relating to the field discovered by Oersted are summarized in the equation

$$\oint \mathbf{H}\, ds = \frac{4\pi}{c} i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the path of integration encircles the current in the right-handed screw sense. For the field of the straight wire, by taking as the path of integration a circle of radius r round the axis of the wire, we thus obtain

$$2\pi r \,|\, \mathbf{H} \,| = \frac{4\pi}{c} i,$$

or

$$|\, \mathbf{H} \,| = \frac{2i}{cr}.$$

Here again, in the same way as we did with Ohm's law, we can pass from the general equation (1) to a differential law, by assuming that (1) holds at all points in the interior of any conductor which is carrying a current. The current flowing across the element of area dS is $i = i_n dS$, so that by applying Stokes's theorem (p. 35) we obtain from (1)

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c} \mathbf{i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

The determination of the magnetic field of a given steady current can be effected in various ways, starting from (1) or (1a), viz. either by direct application of (1) or by the methods—to be discussed immediately—of the *magnetic shell*, the *Biot-Savart law*, or, finally, the *vector potential*.

The first method—*immediate application of* (1)—leads quickly to the result in those cases where something is known about the distribution of the field to begin with, from symmetry or some other circumstance, as e.g. in the case of a *straight wire of circular section*. In this case, let the radius be a, and take concentric circles about the axis as paths of integration. Then

$$2\pi r H = \frac{4\pi}{c} i \quad \text{outside the wire} \quad (r > a),$$

and $$2\pi r H = \frac{4\pi}{c} i \frac{r^2}{a^2} \quad \text{inside the wire} \quad (r < a),$$

so that $$H = \frac{2i}{rc} \quad \text{outside}; \quad H = \frac{2i}{a^2 c} r \quad \text{inside.}$$

Again, in a very long *coil* (*solenoid*) *carrying a current*, we know that practically the whole field is inside the coil, and that it is in the direction of the axis. We therefore choose as the path of integration a small rectangle, two opposite sides of which are parallel to the axis of the coil and 1 cm. long, one of them lying inside and the other outside the coil. For this path $\int \mathbf{H} ds$ is simply equal to the field H within the coil, so long as we are sufficiently far from the ends. If the coil has n turns per centimetre, the current flowing through our rectangle is in. We therefore have within the solenoid

$$H = \frac{4\pi}{c} ni.$$

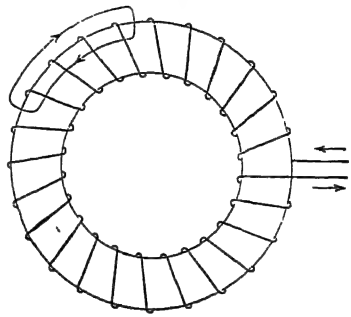
Fig. 2

This result is still practically correct when the wire is coiled round a ring, provided its diameter is small compared with the diameter of the ring (cf. fig. 2, where the path of integration is shown on the left).

The method of the *magnetic shell* (magnetic double stratum) is based on the following remark. If we consider the whole closed path of the current **i**, we see that the integral $\oint \mathbf{H} ds$ is zero when taken over any closed curve which is not linked with the path of the current. If then we construct any surface, subject only to the condition that its edge coincides exactly with the path of the current, $\oint \mathbf{H} ds$ will be zero for any curve which does not pass through this surface or barrier, but equal to $\pm 4\pi i/c$ for any curve which pierces the surface once.

We can therefore also derive the magnetic field of a linear current from a potential ϕ_m, which is subject to the single condition that it changes suddenly by $4\pi i/c$ when we pass through the barrier. It now follows from § 9, p. 31, that *the magnetic field of a current flowing in a closed circuit is identical with that of a homogeneous magnetic shell of moment* $\tau = \mathrm{i}/\mathrm{c}$ *having the circuit for its boundary.*

Hence, for example, when the form of the circuit is a circle of radius a, we can at once assign the field on the axis by utilizing the calculation of the field of a circular disk (p. 28), replacing in equation

(23c) v_z by H and the moment $\omega\eta$ by i/c. At the centre of the circle we find

$$H = \frac{2\pi i}{ca}.$$

On this formula is based the use of the tangent galvanometer, in which the field of the current is compared with that of the earth.

With respect to the *effect at great distances*, it is only the total moment of the magnet ($\mathbf{M} = \int \mathbf{I}\,dV$) which matters, and therefore in a magnetic shell only the integral value $\mathbf{M} = \int \mathbf{n}\tau\,dS$. A plane circuit, which encloses the plane area S, therefore acts at great distances like a permanent magnet of moment

$$\mathbf{M} = \frac{i\mathbf{S}}{c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

The magnetic moment corresponding to a straight coil of n turns, and cross-section S, is accordingly

$$\mathbf{M} = \frac{ni\mathbf{S}}{c}.$$

The *Biot-Savart* law for the determination of the magnetic field of a given current distribution may be obtained by an application of our earlier results on homogeneous double strata (§ 9, p. 32). According to these results the solenoidal field due to such a shell can be represented by the line integral

$$\mathbf{H} = \frac{i}{c}\oint \frac{[d\mathbf{s}\,\mathbf{r}]}{r^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

taken round the boundary curve.

Thus the magnetic field of a closed circuit can be regarded as made up of the sum of contributions from the individual elements of current $i\,ds$ to the magnetic force at the point concerned, in accordance with the rule: the field of an element of current $i\,ds$, at a point the radius vector to which from the element is $\mathbf{r}$, is perpendicular to the plane of ds and $\mathbf{r}$, and is equal to $i\,ds \sin\alpha / cr^2$, where α is the angle between ds and $\mathbf{r}$. The " sense " of $\mathbf{H}$ is defined by the rule that a displacement of ds in its own direction combined with a rotation about ds in the sense corresponding to that of $\mathbf{H}$ gives a right-handed screw motion.

For the field at the centre of a circle of radius a, (2) gives, since ds is perpendicular to the radius,

$$H = \frac{i}{ca^2} 2\pi a = \frac{2\pi i}{ca},$$

which agrees with the value found by the magnetic shell method. The resolution into separate elements of current suggested by the Biot-Savart law (2) is, however, rather an arbitrary procedure, since these elements cannot exist independently.

The *method of the vector potential* follows readily from a simple transformation of the Biot-Savart law (2). The equation (2) can obviously be written in the form

$$\mathbf{H} = -\frac{i}{c}\oint \left[ds,\ \mathrm{grad}_f \frac{1}{r}\right]. \quad . \quad . \quad . \quad . \quad (2a)$$

Hence, taking x, y, z as the co-ordinates of the field point at which $\mathbf{H}$ is to be calculated, and $d\xi$, $d\eta$, $d\zeta$ as the components of ds at the point (ξ, η, ζ), we have, for the rectangular components of $\mathbf{H}$,

$$H_x(x, y, z) = -\frac{i}{c}\oint \left\{\frac{\partial}{\partial z}\left(\frac{1}{r}\right) d\eta - \frac{\partial}{\partial y}\left(\frac{1}{r}\right) d\zeta\right\};$$

for $\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z-\zeta}{r^3} = -\left(\frac{\mathbf{r}}{r^3}\right)_z$, and so on.

We now apply to (2a) the general formula of vector analysis (p. 248)

$$[\mathbf{B}\ \mathrm{grad} f] = f\,\mathrm{curl}\,\mathbf{B} - \mathrm{curl}\,(f\mathbf{B}),$$

which holds for any vector $\mathbf{B}$ and any scalar f.

Since in (2a) the differentiations are with respect to the co-ordinates of the field point only, which do not enter into ds, this gives

$$-\left[ds\ \mathrm{grad}_f \frac{1}{r}\right] = \mathrm{curl}\left(\frac{ds}{r}\right).$$

Hence $$\mathbf{H} = \mathrm{curl}\left(\frac{i}{c}\oint \frac{ds}{r}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This equation exhibits $\mathbf{H}$ as the curl of another vector,

$$\mathbf{H} = \mathrm{curl}\,\mathbf{A}_0,\ \mathbf{A}_0 = \frac{i}{c}\oint \frac{ds}{r}, \quad . \quad . \quad . \quad . \quad (3a)$$

which has already occurred in an earlier section (p. 38), where we called it the vector potential. This form for $\mathbf{H}$, however, is only valid when there are no magnetic materials present. For, in virtue of the identity div curl $= 0$, only a solenoidal vector—which in general $\mathbf{H}$ is not—can be represented as the curl of a vector potential. Consequently in the general theory which is developed later, the vector we derive from a vector potential is not $\mathbf{H}$, but a vector $\mathbf{B}$ not yet introduced, which is always solenoidal.

According to (3a) the contribution of a current element $i\,ds$ to the vector **A** has always the direction of ds. The method of the vector potential is therefore particularly suitable for calculating the field of straight parallel currents, since in that case the direction of **A** is definitely given. As an example of this type, we shall work out the *field of two equal and opposite currents* i *in two parallel wires*. We take the direction of the current in the first wire as the positive x-axis (fig. 3), and calculate A_x at a point whose distances from the wires are r_1 and r_2.

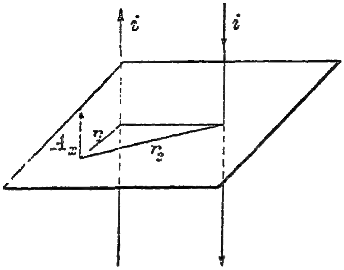

Fig. 3

Clearly A_y and A_z are zero. Suppose first of all that the wires are finite and of length $2L$. Then, by (3a), in the mid plane,

$$A_x = \frac{i}{c}\int_{-L}^{L}\frac{ds}{\sqrt{s^2+r_1^2}} - \frac{i}{c}\int_{-L}^{L}\frac{ds}{\sqrt{s^2+r_2^2}}.$$

If we write for brevity $\eta = s/r$, $\eta_1 = L/r_1$, $\eta_2 = L/r_2$, we find

$$A_x = \frac{i}{c}\cdot 2\int_{\eta_2}^{\eta_1}\frac{d\eta}{\sqrt{1+\eta^2}} = \frac{2i}{c}\log\frac{\eta_1+\sqrt{1+\eta_1^2}}{\eta_2+\sqrt{1+\eta_2^2}}$$

$$= \frac{2i}{c}\left\{\log\frac{\eta_1}{\eta_2} + \log\frac{1+\sqrt{1+1/\eta_1^2}}{1+\sqrt{1+1/\eta_2^2}}\right\}.$$

Now take the limit when L becomes infinite. The second logarithm vanishes, and we have

$$A_x = \frac{2i}{c}\log\frac{r_2}{r_1}.$$

The curves $A_x =$ const., in a plane perpendicular to the wires, are therefore circles with their centres on the line joining the points where the wires cut the plane, and such that these two points are conjugate (inverse) with respect to each circle.

For the magnetic field, we have now

$$H_y = \frac{\partial A_x}{\partial z},\quad H_z = -\frac{\partial A_x}{\partial y}.$$

H has the same numerical value as the gradient of A_x, but the direction of **H** coincides with the direction of the curves $A_x =$ const. The curves $A_x =$ const. therefore give a correct picture of the run of the lines of force in the neighbourhood of the two wires (fig. 4).

In connexion with this result there is a point which is worth noticing. If the two wires carry no current, but are electrically charged, so that the charges per centimetre of their length are $+e$ and $-e$ respectively, they produce an electrostatic potential

$$\phi = \frac{2e}{K} \log \frac{r_2}{r_1},$$

where K is the dielectric constant of the medium surrounding the wires. Hence, except for a numerical factor, ϕ is identical with A_x.

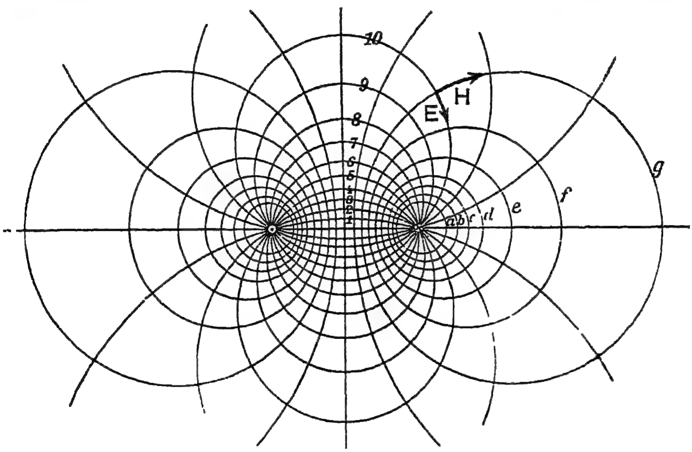

Fig 4.

The same point will come up again later in a more general form when we are dealing with waves in wires.

3. Magnetization and Magnetic Susceptibility.

So far we have considered two ideal limiting cases of a magnetic field, namely the field produced by permanent magnets (materials with assigned magnetization $\mathbf{I}$) in the absence of electric currents, and the field produced by steady currents in the absence of magnetizable materials. The results for these two cases were as follows:

(i) Permanent magnets ($\mathbf{i} = 0$): $\operatorname{div} \mathbf{H} = -4\pi \operatorname{div} \mathbf{I}$; $\operatorname{curl} \mathbf{H} = 0$.

(ii) Steady currents ($\mathbf{I} = 0$): $\operatorname{curl} \mathbf{H} = \frac{4\pi}{c} \mathbf{i}$; $\operatorname{div} \mathbf{H} = 0$.

From these results we deduce the following general equations for the calculation of the field of an arbitrary distribution of steady current $\mathbf{i} = \mathbf{i}(x, y, z)$, combined with magnetized materials ($\mathbf{I} = \mathbf{I}(x, y, z)$) also distributed in any way. The *sources* of $\mathbf{H}$ are the same as those

of $-4\pi\mathbf{I}$; the curl of $\mathbf{H}$ is equal to $4\pi\mathbf{i}/c$. In any region where $\mathbf{i}$ and div $\mathbf{I}$ are both zero, $\mathbf{H}$ is irrotational and solenoidal. At surfaces of discontinuity of $\mathbf{I}$ we have to take of course the surface divergence (discontinuity of the normal component), corresponding to the transition from a continuous, rapidly varying change to the limiting case of an instantaneous finite change.

If then the current distribution $\mathbf{i}$ and the magnetization $\mathbf{I}$ were given in advance, the determination of the field $\mathbf{H}$ would be an equivalent problem to that of calculating a vector field from its vortices and sources, a problem which we have already (p. 37) completely solved. In the present case the equations corresponding to (36*a*) and (36*b*) of p. 37 run

$$\operatorname{div} \mathbf{H} = -4\pi \operatorname{div} \mathbf{I}, \quad \operatorname{curl} \mathbf{H} = \frac{4\pi}{c}\mathbf{i}. \quad . \quad . \quad (4)$$

In point of fact the circumstances here are very much more complicated, on account of the fact that the magnetization itself essentially depends upon the magnetic force. In most cases it is only by the field that it is produced at all. This connexion between $\mathbf{H}$ and $\mathbf{I}$ is a specific property of the material we happen to be dealing with, the property by which its magnetic character is determined. If we group all known materials according to their magnetic behaviour, we get the following classification.

(*a*) *The magnetization is proportional to the field:*

$$\mathbf{I} = \kappa\mathbf{H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

The factor κ is called the *magnetic susceptibility* per unit volume; it is independent of $\mathbf{H}$, but may vary with the temperature.

Among materials characterized by the simple relation (4*a*), the two following types can be distinguished.

(*a*1) *Diamagnetic bodies.*—For these κ is negative, and independent of the temperature; it is numerically a very small proper fraction. Examples of its values are:

Hydrogen	$\kappa = -0{\cdot}5 \times 10^{-9}$
Water	$-0{\cdot}77 \times 10^{-6}$
Gold	-3×10^{-6}
Bismuth	-14×10^{-6}.

In diamagnetic substances $\mathbf{I}$ is therefore opposite in direction to the field $\mathbf{H}$. We can explain diamagnetism *qualitatively* if we assume that within the individual atoms there are electric circuits without resistance. When an external magnetic field comes into play, currents will be induced in these circuits in accordance with the general laws of induction, and the magnetic moment of these currents (cf. (1*b*),

p. 128) will be opposite to **H** in direction. A *quantitative* explanation can be given by the theory of electrons, which will be dealt with in Vol. II.

Diamagnetism is a general property of matter, and accordingly is present in all substances. It is, however, so small that in practice it escapes observation whenever the material concerned is *also* paramagnetic or ferromagnetic.

(*a*2) *Paramagnetic bodies.*—In these κ *is positive and as a rule inversely proportional to the absolute temperature* (Curie's law):

$$\kappa = \frac{C}{T}. \qquad (4b)$$

The following are some observed values at room temperature:

Oxygen	$\kappa =$	$0{\cdot}14 \times 10^{-6}$
Platinum		29×10^{-6}
Manganese		300×10^{-6}.

To picture the action of paramagnetism, we must suppose that the individual molecules possess a fixed magnetic moment, and that these elementary magnets are partly straightened out by an external field. The orienting action of the field acts against the irregular temperature motion. This has a bearing on Curie's law, for the same field is able to produce a greater degree of regular arrangement at low temperatures than at high.

(*b*) *The magnetization is not proportional to the magnetic force.*—This class consists essentially of the ferromagnetic materials iron, cobalt, nickel, and the Heusler alloys. The magnetic behaviour of these materials is very complicated, and depends to a large extent on circumstances which are often apparently trivial. We must therefore be content with a very general classification of properties. The most striking characteristic of the ferromagnetic class of substances is the high value of the magnetic moment for a given magnetic force. (It is frequently more than a million times as great as in other substances.) Again, **I** does not in this class change linearly with **H**; on the contrary, for relatively low applied fields, easy to produce in practice, a condition of *saturation* is reached. The saturation values (I_∞) of the magnetization, which cannot be much exceeded even with very strong fields, run about as follows:

Iron	$4\pi I_\infty$	= 22,000 gauss
Nickel		= 6,000 ,,
Cobalt		= 18,000 ,,

These values are nearly independent of the state of the material, as resulting from mechanical and heat treatment, and also of small

admixtures of chemical impurities. On the other hand, the "magnetization curve", i.e. the curve showing how **I** changes as **H** increases, depends in the most pronounced way on the special treatment to which the specimen has previously been subjected. Here again we can distinguish two extreme cases, as under.

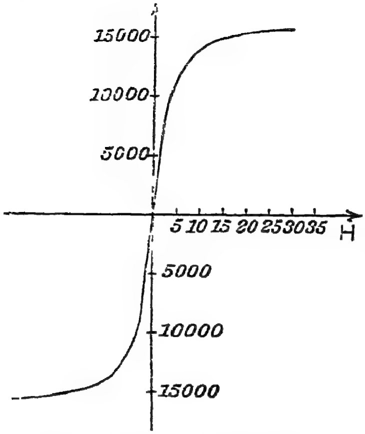

Fig. 5*a* (Ordinates represent 4π**I**)

(*b*1) *Magnetically soft substances* are those in which **I** is still at least a one-valued function of **H**. The graph of this function has in typical cases the general form shown in fig. 5*a*—rising steeply at first, then getting flatter and flatter, till finally (at saturation) it becomes practically horizontal. Since the slope of the curve is almost straight at first, we can speak of an "initial susceptibility", which we may define either as the quotient $|\mathbf{I}|/|\mathbf{H}|$ of the values concerned, or, seeing that the curve is so steep, as $\partial|\mathbf{I}|/\partial|\mathbf{H}|$. Its value for different kinds of iron is in round numbers between 50 and 1000. Perfectly soft, i.e. absolutely reversible, ferromagnetic substances could scarcely be expected to occur in nature. Properly speaking, there are only "softer" or "harder" substances, corresponding to the smaller or greater breadth of the hysteresis loop (see below).

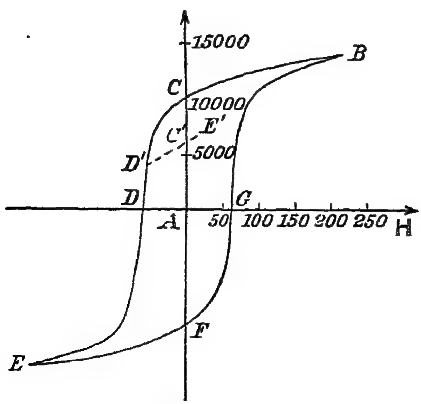

Fig. 5*b* (Ordinates represent 4π**I**)

(*b*2) *Magnetically hard substances.*—In these **I** is not a one-valued function of **H** at all, the magnetization definitely depending also on the field strengths to which the specimen has previously been exposed. The typical course of the magnetization curve is shown in fig. 5*b*. If we subject the sample, which is not magnetized to begin with, to an increasing field **H**, **I** goes through values represented by an arc AB, which qualitatively does not differ essentially from the curve (fig. 5*a*) for a magnetically soft substance. But if we now allow **H** to

become smaller again, **I** begins by diminishing much more slowly than it formerly increased (arc BCDE). For the field $\mathbf{H} = 0$ we have still a "residual magnetization" of amount AC. This quantity we shall denote later by $\mathbf{I}_0$. To bring the magnetization to the value zero we must apply the "coercive force" AD in the direction opposed to **I**. Residual magnetization and coercive force furnish a measure of the magnetic hardness of the substance. For large negative values of **H** we reach saturation again at E. From there, with suitable changes of the value of **H**, **I** goes back through the arc EFGB, thus closing the "hysteresis loop". If the field is then repeatedly altered backwards and forwards from saturation in the one direction to saturation in the other, **I** always goes over practically the same loop.

Quite a different result is obtained, however, if we only go as far as some definite point of the loop, say D′, and then allow **H** to increase again. We then get, for not too great an increase of **H**, almost a straight line, such as the dotted straight line D′C′E′ in fig. 5*b*, which can now be described again backwards without change. If then in all further applications of magnetic force to the material we remain between the limits D′ and E′, we can within those limits speak of a reversible magnetization, and characterize the material by a "magnetic equation of state",

$$\mathbf{I} = \kappa' \mathbf{H} + \mathbf{I}_0', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4c)$$

where therefore κ' denotes the gradient of the straight line D′E′, and $\mathbf{I}'_0$ the segment AC′.

In the interior of a permanent magnet, in the absence of currents and other magnets, the field **H** has substantially the opposite direction from the magnetization. Such a magnet is therefore situated on the part CD of the hysteresis curve, and can be represented, for example, by the point D′ already considered. In individual cases, the field in the interior of a permanent magnet, with given magnetization, depends also on the form of the magnet.

For moderately small changes in **H**, e.g. changes due to alteration of the air gap in an almost closed ring magnet, the corresponding change in **I** can therefore be obtained from the line D′E′ or from equation (4*c*).

All ferromagnetic materials show the property emphasized by the name only so long as their temperature remains under a certain value Θ, which is characteristic of the material in question, and is called the Curie point. For iron the Curie point is 774° C., for nickel 372° C., and for cobalt 1131° C. Above their Curie point all ferromagnetic materials show normal paramagnetism, with this difference, however, that in the Curie law (4*b*) the absolute temperature T has to be replaced by the distance $T - \Theta$ from the Curie point:

$$\kappa = \frac{C}{T - \Theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4d)$$

(Curie-Weiss law).

4. Magnetic Induction.

Without making any assumption about the constitution of the material, from the point of view of the possibilities just explained, we can give equations (4), p. 132, another form by introducing a vector $\mathbf{B}$, called the *magnetic induction*, which we define by the equation

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

While $\mathbf{E}$ is determined by its sources and vortices, equations (4), p. 132, show that $\mathbf{B}$ is always solenoidal, and is accordingly characterized by its curl alone. We have, in fact,

$$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{curl} \mathbf{B} = \frac{4\pi}{c}\mathbf{i} + 4\pi \operatorname{curl} \mathbf{I}. \quad . \quad . \quad . \quad (6)$$

Since $\mathbf{B}$ is solenoidal, it can always be represented as the curl of a vector potential $\mathbf{A}$,

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

the vector potential $\mathbf{A}$ being subjected to the supplementary condition

$$\operatorname{div} \mathbf{A} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6b)$$

It follows at once from (6*a*) and (6*b*), by applying the result of § 11, p. 37, that

$$\mathbf{A} = \frac{1}{c}\int \frac{\mathbf{i} + c \operatorname{curl} \mathbf{I}}{r} dV. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

As in the preceding section, so here also the point must be emphasized that the practical applicability of equation (7) is considerably restricted by the circumstance that $\mathbf{I}$ is not known to begin with, but itself depends in a complicated way on $\mathbf{B}$ or $\mathbf{H}$.

In principle it is immaterial whether, in order to determine the field in a concrete case, we first calculate $\mathbf{H}$ by (4), p. 132, or $\mathbf{B}$ by (6). After we have brought in equation (5) connecting $\mathbf{B}$ and $\mathbf{H}$, the result in the two cases must be the same.

To illustrate the relations involved, we shall consider qualitatively the field of a circular cylinder homogeneously polarized in the direction of its axis, the cylinder consisting of ideally hard magnetic material. If the axis of the cylinder is parallel to the x-axis, then the data are: $I_x = \text{const.} = I$ within the cylinder, $I_x = 0$ outside; $I_y = I_z = 0$ everywhere. Let it be expressly emphasized that we are here considering an ideal case, not realized in practice. Actually, $\mathbf{I}$ always depends upon $\mathbf{H}$; this is brought out in the figures by the refraction of the lines of force (fig. 6*a*) at the end faces, and of the lines of induction (fig. 6*b*) at the curved face. In fig. 5*b* our idealization would be represented by the straight line D′E′ (reversible magnetization) be-

coming horizontal. The divergence of **I** is concentrated on the two end faces, where there is a surface divergence of amount $\pm$ **I**. Its curl, on the other hand, is concentrated into a surface curl on the curved face, where **I** jumps from I to 0. From equations (4) and (5) we gather the following description of the fields **H** and **B**. The normal component of **H** changes suddenly by $\pm 4\pi I$ when we pass through an end; for the orientation assumed, H_x therefore changes by $+ 4\pi I$ in both cases when we pass from the interior through an end face. Everywhere else **H** is irrotational and solenoidal; in particular it is continuous for passage through the curved surface.

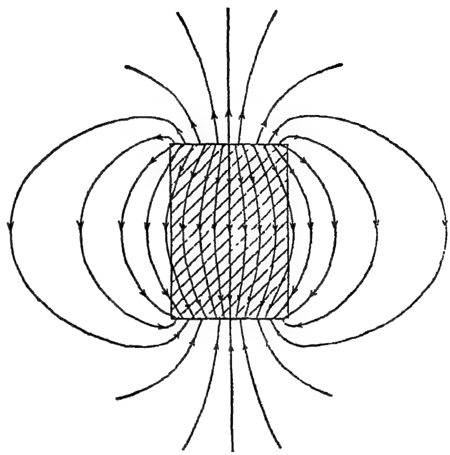

Fig. 6*a*

These data are sufficient for the unique specification of **H**. The lines of force thus obtained are shown graphically in fig. 6*a*. *Within* the cylinder the direction of **H** is, broadly speaking, opposite to that of **I**; for a long cylinder, it is approximately equal, near the ends, to $2\pi I + IS/l^2$; where S is the area of the section, and l is the length of the cylinder. Here it is assumed that S is small compared with l^2. ($2\pi I$ is the contribution from the end itself, IS/l^2 the Coulomb force due to the other end.) Immediately *outside* the end faces, **H** has practically the same direction as **I**, and the value $2\pi I - IS/l^2$. In the neighbourhood of the curved surface, **H** is inclined at an angle to the direction of **I**. The curved surface, as a boundary, is ignored by the lines of force.

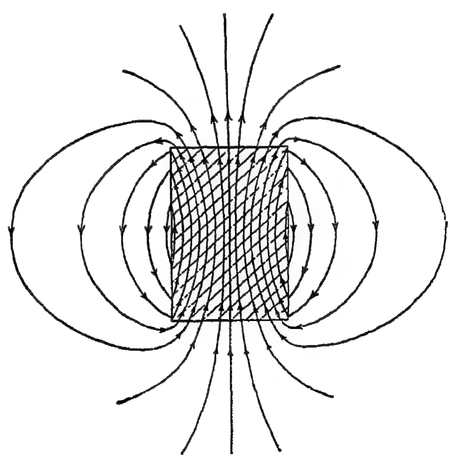

Fig. 6*b*

We consider next the run of the lines of induction **B**. Outside the

cylinder, **H** and **B** are of course identical. In the interior, however, we must add the vector $4\pi\mathbf{I}$ to **H** throughout. On doing so, we obtain fig. 6*b* as the diagram for the lines of induction. Its general features are governed by the surface curl of **I** concentrated on the curved surface.

The diagrams (figs. 6*a* and 6*b*), showing the vectors **H** and **B**, are entirely equivalent as representations of the facts. Still, on looking at the two figures one can scarcely refrain from putting the question: which of the two descriptions is the more natural? The answer to this question depends altogether upon the idea which we form of the nature of atomic magnetism. If we look upon the individual atoms of the magnet as little bar magnets with a north and a south pole, we will naturally be led to the picture of fig. 6*a*: the little magnets, straightened out in the direction of the axis of the cylinder, produce on the end faces a surplus of positive or negative "free" magnetism, which acts as a source or sink for the lines of force. Very different will the answer be, if (following a hypothesis first put forward by Ampère) we regard the atoms, not as little bar magnets, but as minute closed currents, which in fact (p. 128) act as small magnets perpendicular to the planes of the circuits. If these elementary currents have their axes in the direction of the axis of the cylinder, they will certainly neutralize each other in the interior, but there will be left over on the curved surface a finite, superficially distributed current encircling the cylinder (cf. e.g. fig. 12, p. 35). The field of the vector **B** in fig. 6*b* presents itself as the immediate consequence of this surface current, **B** being defined by its curl, a vector which coincides with the surface current. At the present day we know that Ampère's hypothesis is essentially correct. (Its elementary currents are interpreted in the electron theory as convection currents, due to the motion of the electrons.) To the question suggested above we can therefore make the perfectly definite answer, that fig. 6*b* is the figure which fits the natural character of magnetism. Only on the curved surface of the cylinder is there "actually" something present, namely the "free" current $\mathbf{i}' = c \operatorname{curl} \mathbf{I}$ which, according to (6), p. 136, along with the conduction current $\mathbf{i}$ defines the curl of **B**. *Not the magnetic force* **H**, *but the induction* **B**, is the primary magnitude. The vector $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{I}$, like its sources as displayed in fig. 6*a*, must be regarded as purely artificial, only employed for greater convenience in the statement of the formulæ.

The general relation (5), p. 136, connecting **B** and **H** can be put in a simpler form if **I** is known as a function of **H**. *First, if* **I** *is proportional to* **H**, or

$$\mathbf{I} = \kappa\mathbf{H},$$

then

$$\mathbf{B} = \mu\mathbf{H}, \qquad (8)$$

where

$$\mu = 1 + 4\pi\kappa. \qquad (8a)$$

The name *magnetic permeability* is given to μ. It is therefore less than 1 for diamagnetic, and greater than 1 for paramagnetic materials. In both, however, μ differs only very little from 1 (less than 1 in 1000). Only in ferromagnetic materials is μ decidedly greater than 1; in these, at the steepest part of the curve (5*a*), p. 134, it can reach values between 1000 and 10,000. For permanent magnets, subjected to reversible changes of state (D′C′E′ in fig. 5*b*), within the narrow range of $\mathbf{H}$ for which this is possible, we can express $\mathbf{B}$, by (4*c*), p. 135, in the form

$$\mathbf{B} = \mu'\mathbf{H} + 4\pi\mathbf{I}_0, \quad . \quad . \quad . \quad . \quad . \quad (8b)$$

of which we shall make use later in our discussion of the energy of the magnetic field.

5. Faraday's Law of Induction.

In the year 1831 Faraday made the fundamental discovery that an electric current is generated in a closed conducting circuit (a loop of wire, for example) when a magnet in its neighbourhood is moved. Closer experimental investigation of the phenomenon led to the following quantitative result with regard to the current so arising.

Let R be the ohmic resistance of the circuit, S a surface having the circuit as its bounding edge but otherwise arbitrary. We choose a definite currency ds for the circuit, and accordingly a definite direction for the normal to the surface, in accordance with the right-handed screw rule. The current i is counted positive or negative according as it flows in the sense of ds, or the opposite. Then the law of induction for the current in Faraday's fundamental experiment runs:

$$iR = -\frac{1}{c}\frac{d}{dt}\iint B_n\, dS. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The product of resistance and current strength is at any one moment equal to the quotient by c *of the time rate of diminution of the flux of induction through a surface bounded by the circuit.*

For the quantity

$$-\frac{1}{c}\frac{d}{dt}\iint \mathbf{B}_n\, dS$$

the conveniently short expression "magnetic decay"* will occasionally be used. It is all the same whether this "decay" is due to the field changing with the time while the circuit remains at rest, or to the circuit moving while the field remains constant. The result (9) provides us with an entirely new method, of great practical impor-

* Ger. *Magnetischer Schwund.*

tance, for the *exploration of a given magnetic field.* For this purpose we take a search coil of dimensions small enough to justify us in regarding the field in its neighbourhood as homogeneous. The coil is connected to a ballistic galvanometer. So long as the coil is at rest in a constant field **B**, the galvanometer shows no current. The flux of induction through the effective area S (area $\times$ number of turns) of the coil is $B_n S$. If we now withdraw the coil from the field to a place where there is no field, then during the motion a current flows in the coil of amount

$$i = -\frac{1}{cR}\frac{d}{dt}B_n S.$$

The total quantity of electricity, which the ballistic galvanometer directly indicates if the motion of the coil is sufficiently rapid, is therefore

$$e = \int_0^t i\,dt = \frac{B_n S}{cR}.$$

Hence the throw of the galvanometer in the experiment considered measures directly the component B_n, perpendicular to the plane of the coil, of the induction at the place occupied by the coil before it was withdrawn.

Another way of making the experiment is to leave the coil in its place, but to turn it through an angle of 180° round an axis in its plane (earth inductor). In that case B_n changes sign, and we find for e double the value given above.

We shall now express the law of induction (9) in a more general form, by eliminating the current strength i from it by means of Ohm's law. For this purpose, we shall in the first place so far generalize the law (9) as to suppose that in the circuit considered there is also an applied E.M.F. $E^{(e)}$ acting. This E.M.F. will itself give rise to a current $E^{(e)}/R$, so that (9) must be replaced by

$$iR = E^{(e)} - \frac{1}{c}\frac{d}{dt}\iint B_n\,dS. \quad . \quad . \quad . \quad . \quad (9a)$$

But by Ohm's law, in the differential form in which we shall always use it, we have

$$\mathbf{i} = \sigma(\mathbf{E} + \mathbf{E}^{(e)}),$$

i.e. the current strength at any point is to be defined solely by the combined action of the electric intensity and the impressed forces at the point in question. But we shall then have, after integration over the volume of the linear conductor (as at p. 121),

$$iR = E^{(e)} + \oint \mathbf{E}_s\,ds.$$

Now in the electrostatic field the second term here was always zero, since **E** was irrotational, but in the present case comparison with (9a)

shows that when the flux of induction is changing we must always have

$$\oint \mathbf{E}_s ds = -\frac{1}{c}\frac{d}{dt}\iint B_n dS; \quad . \quad . \quad . \quad . \quad (10)$$

or the total E.M.F. in the circuit is equal to the rate of decrease of the flux through it, divided by c.

In (10), the law of induction (9) is expressed in a form from which the resistance R, a constant of the wire, has completely disappeared. Equation (10) refers in the first instance to the loop of wire, but it admits of a remarkable generalization, which is fundamental for all that follows. We assert in fact that the truth of the relation (10) is quite independent of the presence of the wire; in other words, that the total E.M.F. *round any closed curve whatever* is correctly given by (10). The assertion can in the first place be justified for the case when the path of integration is not along the wire itself, but along a curve immediately adjacent to it in empty space. For, on account of the continuity of the tangential components of $\mathbf{E}$, the value of $\oint E_s ds$ is not altered by this displacement of the path. However, this new reading of equation (10) is in its complete generality a hypothesis, which we must justify by testing its consequences.

This new view allows us to pass at once to a *differential form of the law of induction*; for, if (10) holds for any surface element, however situated, we have only to apply Stokes's theorem to obtain a differential relation connecting the vectors $\mathbf{E}$ and $\mathbf{B}$.

In media at rest, the flux of induction only changes so far as the vector $\mathbf{B}$ changes. Hence we can differentiate under the sign of integration on the right side of (10), and Stokes's theorem then gives at once

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{dt}, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

for media at rest.

But when the body, in which $\mathbf{E}$ is to be calculated, is moving with the velocity $\mathbf{u}$, the time rate of change of the flux of induction is to be taken, according to the expression for it in (9), for a surface S moving with the material. Now, by equation (37a), p. 40, we have in all cases when the element of area dS is moving with velocity $\mathbf{u}$

$$\frac{d}{dt}\iint B_n dS = \iint \left\{ \frac{\partial B_n}{\partial t} + u_n \operatorname{div} \mathbf{B} - (\operatorname{curl} [\mathbf{uB}])_n \right\} dS.$$

Since $\operatorname{div} \mathbf{B} = 0$ here, we accordingly obtain

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\left(\frac{\partial \mathbf{B}}{\partial t} - \operatorname{curl} [\mathbf{uB}]\right), \quad . \quad . \quad . \quad (11a)$$

for moving media.

This equation is frequently written in an abbreviated form, by introducing the special kind of time differentiation explained in § 12, p. 39:

$$\underline{\dot{\mathbf{A}}} \equiv \dot{\mathbf{A}} + \mathbf{u} \operatorname{div} \mathbf{A} - \operatorname{curl} [\mathbf{uA}].$$

Equation (11a) then becomes

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c} \underline{\dot{\mathbf{B}}}.$$

If we go back again from (11a) to the total E.M.F. in a wire loop, we see in the two terms on the right-hand side the two possible causes of "magnetic decay"; first, the time rate of change of $\mathbf{B}$, the only cause acting when the wire is at rest; secondly, the motion of the wire, which is the sole cause of the "decay" when the field of induction is constant in time. It is easy to see that $-d\mathbf{s}[\mathbf{uB}]\,dt$, i.e. $\mathbf{B}[\mathbf{u}\,d\mathbf{s}]\,dt$ represents the flux of induction across the element of area $[\mathbf{u}\,dt,\ d\mathbf{s}]$ swept out by the boundary curve in time dt.

It should be expressly mentioned that the field equations for moving bodies are in reality essentially more complicated than equation (11a), which only represents an approximation, but one which is amply sufficient for all practical applications. To derive the exact form of the equations for arbitrary values of $\mathbf{u}$, we require the theory of electrons and the theory of relativity. A detailed discussion will be given in Vol. II.

CHAPTER VIII

Electrodynamics of Media at Rest

1. Maxwell's Equations for Bodies at Rest.

We are now in a position to set forth Maxwell's equations for bodies at rest in their final form. It is true that equation (1*a*), p. 126,

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c}\,\mathbf{i},$$

defining the magnetic field of a steady distribution of current, still requires for one case an essential complement of critical import. The case is that in which the currents are not closed, but begin and end, for example, at the coatings of a condenser. At such places the divergence of $\mathbf{i}$ is not zero, while the left side of the above equation is always solenoidal (div curl $\equiv 0$). To obtain an equation which is valid in all cases, we must therefore either find an entirely new relation or else make the right side of the equation also solenoidal by adding another vector to it. Maxwell chose the latter course. Thus, as has already been indicated in § 2, p. 113, a source of $\mathbf{i}$ necessarily implies diminishing density of charge at the point concerned; in fact, we have by Gauss's theorem

$$\operatorname{div} \mathbf{i} = -\frac{\partial \rho}{\partial t}.$$

On the other hand, charge density is equivalent to divergence of the displacement vector $\mathbf{D}$:

$$4\pi\rho = \operatorname{div} \mathbf{D}.$$

Hence we have

$$\operatorname{div} \mathbf{i} = -\frac{1}{4\pi}\operatorname{div}\frac{\partial \mathbf{D}}{\partial t}.$$

This equation states that the vector

$$\mathbf{c} = \mathbf{i} + \frac{1}{4\pi}\dot{\mathbf{D}}$$

is always solenoidal. *The required complement of the conduction current* $\mathbf{i}$ *is therefore found. It is the displacement current*

$$\frac{1}{4\pi}\dot{\mathbf{D}} = \frac{1}{4\pi}\dot{\mathbf{E}} + \dot{\mathbf{P}},$$

the introduction of which into the fundamental equations forms the kernel of the whole Maxwellian theory. This is really the only point—but it is one of decisive importance—in which the equations of Maxwell's theory substantially differ from those of the older action at a distance theory.

To equation (1*a*), p. 126, as now extended, we add three more, namely the law of induction (11), p. 141, and the two equations (3*a*), p. 75, and (6), p. 136, involving the divergence of **D** and of **B**. We have therefore the *four fundamental equations*

$$\text{I.} \qquad \operatorname{curl} \mathbf{H} = \frac{4\pi}{c}\,\mathbf{i} + \frac{1}{c}\,\dot{\mathbf{D}},$$

$$\text{II.} \qquad \operatorname{curl} \mathbf{E} = -\frac{1}{c}\,\dot{\mathbf{B}},$$

$$\text{III.} \qquad \operatorname{div} \mathbf{D} = 4\pi\rho,$$

$$\text{IV.} \qquad \operatorname{div} \mathbf{B} = 0,$$

as the final expression of Maxwell's theory for bodies at rest. To obtain a complete system from these equations, we have to add to them three others, connecting the three vectors **i**, **D**, **B** with the electric and magnetic force vectors **E** and **H**. It is only when **i**, **D**, and **B** can be eliminated from equations I to IV by means of these supplementary equations that the state of the system at any time is uniquely defined when its initial state is arbitrarily given. In their simplest form the three supplementary equations are the following:

$$\text{V.} \qquad \mathbf{i} = \sigma(\mathbf{E} + \mathbf{E}^{(e)}),$$

$$\text{VI.} \qquad \mathbf{D} = K\mathbf{E},$$

$$\text{VII.} \qquad \mathbf{B} = \mu\mathbf{H}.$$

These are the equations for an *isotropic* body which is *not ferromagnetic*, and in which the conductivity is σ, the dielectric constant K, and the magnetic permeability μ. All three equations V, VI, and VII are therefore determined by the special properties of the material present in the field. As may therefore be expected, they never hold with the rigour and generality which can be claimed for equations I to IV, except in a vacuum, where we have exactly $\sigma = 0$, $K = 1$, $\mu = 1$.

Quite apart from the fact that VII fails altogether for ferromagnetic substances, there are also phenomena such as dielectric "after-effect", and "residual charge" in Leyden jars, which are completely ignored in VI. Moreover, VI fails in the case of rapidly alternating fields (light waves), for which, as experiment shows, K becomes a function of the frequency of the field, so that we can hardly continue to speak of K as a dielectric constant. The interpretation and theoretical calculation of the quantities σ, K, μ which have been introduced here as constants

of the material, will receive detailed consideration later in the theory of electrons.

The integral of energy for Maxwell's equations.—If we multiply I by $-\mathbf{E}$, II by $\mathbf{H}$, and add, we find

$$\mathbf{H} \operatorname{curl} \mathbf{E} - \mathbf{E} \operatorname{curl} \mathbf{H} + \frac{4\pi}{c} \mathbf{i}\mathbf{E} = -\frac{1}{c} \mathbf{E}\dot{\mathbf{D}} - \frac{1}{c} \mathbf{H}\dot{\mathbf{B}}.$$

We now use the identity (p. 36)

$$\mathbf{H} \operatorname{curl} \mathbf{E} - \mathbf{E} \operatorname{curl} \mathbf{H} = \operatorname{div} [\mathbf{E}\mathbf{H}],$$

and, after integration over any volume and multiplication by $c/4\pi$, obtain

$$-\frac{1}{4\pi}\int(\mathbf{E}\dot{\mathbf{D}} + \mathbf{H}\dot{\mathbf{B}})\, dV = \int(\mathbf{i}\mathbf{E})\, dV + \frac{c}{4\pi}\int[\mathbf{E}\mathbf{H}]_n\, dS. \quad (1)$$

This equation depends on the field equations I to IV only, and these are rigorously accurate; we must therefore consider (1) also as exactly correct for bodies at rest.

We shall only discuss (1) here for the case when the supplementary equations V, VI, and VII are satisfied. We then have

$$-\frac{d}{dt}\left\{\int\left(\frac{K}{8\pi}\mathbf{E}^2 + \frac{\mu}{8\pi}\mathbf{H}^2\right) dV\right\}$$
$$= \int\frac{\mathbf{i}^2}{\sigma}\, dV - \int\mathbf{i}\mathbf{E}^{(e)}\, dV + \int\frac{c}{4\pi}[\mathbf{E}\mathbf{H}]_n\, dS. \quad (1a)$$

We read this equation as follows: *the electromagnetic field possesses the energy density*

$$u = \frac{1}{8\pi}(K\mathbf{E}^2 + \mu\mathbf{H}^2). \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

If the total energy $U = \int u\, dV$ contained in the volume V diminishes, then according to (1a) three different items of energy may make their appearance as the equivalent of the loss. There is first the irreversible Joule heat $\mathbf{i}^2/\sigma$, along with the work done against the impressed forces. These two items together we call the thermochemical activity of the field. They are represented in (1) by the single term ($\mathbf{i}\mathbf{E}$). (Cf. equation (6b), p. 151.)

Again, as a further cause of decrease of the energy of the field, we have in (1a) the surface integral

$$\int N_n\, dS, \text{ where } \mathbf{N} = \frac{c}{4\pi}[\mathbf{E}\mathbf{H}]. \quad . \quad . \quad . \quad . \quad (2)$$

The principle of the conservation of energy therefore requires that there should be a *stream of energy* $\mathbf{N}$ (*per sq. cm. per second*) *across the surface of the region considered.* The vector $\mathbf{N}$ representing this flow of

energy is called the *Poynting vector*. It will occupy our attention in detail in the theory of electromagnetic waves. It should be expressly emphasized, and is inherent in the above method of obtaining (1), that it is only when it is taken over a *closed* surface that the integral $\int N_n \, dS$ has the physical signification of a flow of energy outwards from the region enclosed by the surface.

The vector $\mathbf{N}$ itself may very well have a value other than zero, without there being any noticeable transport of energy. We have only to think of the case where an electrostatic field is crossed by a magnetic field. Here certainly $\mathbf{N}$ can take values as great as we please, nevertheless div $\mathbf{N}$ will always be zero, so that $\mathbf{N}$ can have no effect on the energy balance.

The form (1) of the theorem of energy holds for bodies at rest only. It contains therefore no term referring to mechanical work, of the kind fully considered above for the case of the electric field, and to be separately investigated in next section for the case of the magnetic field.

In Chap. IX, on quasi-steady currents, we shall continue to neglect, in comparison with the conduction current $\mathbf{i}$, the displacement current $\dot{\mathbf{D}}/4\pi$ which is characteristic of Maxwell's theory. For sufficiently slow changes of the field this is certainly permissible. As will appear later, the alternating currents of electrical engineering can from this point of view be considered as only slowly varying. The results which can be obtained in this way are of course exclusively those which were within the reach of the pre-Maxwell action at a distance theory. As we shall see later, when we take the displacement current into account we are led to a finite velocity of propagation of electromagnetic disturbances. According to I, p. 144, to neglect $\dot{\mathbf{D}}/4\pi$ means the same thing as to put div $\mathbf{i} = 0$, i.e. to assume that the currents are quasi-steady. It may therefore be expected that neglect of $\dot{\mathbf{D}}/4\pi$, in the case of currents which vary with the time, is justified whenever the time which the currents take to change sensibly is great compared with the time needed for the electromagnetic disturbances to traverse the distance from one end of the apparatus to the other.

The displacement current does not become important until rapidly varying processes are considered. The special features and full capabilities of Maxwell's theory will therefore not display themselves until we are discussing electromagnetic waves.

2. Energy and Maxwell's Stresses in the Magnetic Field.

We have already justified the expression $(\mathbf{ED})/8\pi$, which was assumed for the energy density of the electrostatic field, by proving (p. 85) that the work done in a displacement of the material in the field is equal to the diminution of the quantity

$$U_{el} = \frac{1}{8\pi} \int (\mathbf{ED})\, dV.$$

The proof required that we should have

$$\mathbf{D} = K\mathbf{E},$$

with the dielectric constant K independent of $\mathbf{E}$. But in the magnetic field, at least when permanent magnets are present, such proportionality of the corresponding vectors $\mathbf{B}$ and $\mathbf{H}$ is out of the question. We can hardly expect then that the energy of the magnetic field should in general be given by the expression analogous to U_{el}, viz. $\int (\mathbf{HB})\, dV/8\pi$. That this expression cannot possibly be correct we recognize at once from the fact that its value for a field of arbitrary permanent magnets (with no currents) is always zero. For in that case curl $\mathbf{H} = 0$, which, along with div $\mathbf{B} = 0$, is sufficient to ensure the vanishing of the integral $\int (\mathbf{HB})\, dV$ (cf. (36m), p. 39).

We must therefore search for a general expression for the magnetic energy U_m, but in doing so shall still confine ourselves to bodies in which changes in the induction are uniquely given by the changes in the magnetic force. We consider first the following elementary case: let a bar of any material, of cross-section S and length l, be bent to a circle, and a conducting wire of resistance R wound uniformly on it with n turns per cm. Let a current i be maintained in the wire by a battery of electromotive force E. Then the battery in time dt does work of amount $W = Ei\, dt$. If the induction B changes in this time to $B + dB$, Faraday's law of induction gives

$$iR = E - \frac{Snl}{c} \frac{dB}{dt}.$$

Further, the magnetic field $\mathbf{H}$ in the ring is determined by $\mathbf{i}$ alone:

$$H = \frac{4\pi}{c} ni.$$

Hence

$$W = Ei\, dt = i^2R\, dt + i\frac{Snl}{c}\, dB$$
$$= i^2R\, dt + \frac{Sl}{4\pi}\, \mathbf{H}\, d\mathbf{B}.$$

But Sl is the volume of the bar. Thus we have the result that, as the equivalent of the work done by the battery, besides the Joule heat there is also produced a quantity of energy $\mathbf{H}\, d\mathbf{B}/4\pi$ per unit volume of the rod. This quantity must be regarded as change of the energy of the rod. The result agrees with the expression $\mathbf{H}\dot{\mathbf{B}}/4\pi$ found in (1), p. 145, for the rate of change of the magnetic energy density. In point of fact the simple example just considered is only a special case of the general method by which we obtained the equation of energy (1).

Strictly speaking, it is change in the free energy that we have before us here since we have tacitly assumed that the process concerned is isothermal (cf. the remark at p. 85). We shall return to the point in Part IV, but for the purposes of this chapter it may be ignored.

We now assume that the induction **B** is known as a function of **H**, say from a magnetization curve like that shown in fig. 1. The result just found suggests that we should define the density of the magnetic energy as

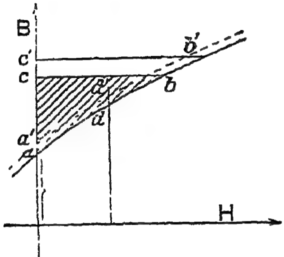

Fig. 1

$$U_m = \frac{1}{4\pi}\int_{\mathbf{H}=0}^{\mathbf{B}} \mathbf{H}\, d\mathbf{B},$$

which would correspond to the shaded area *adbc* in fig. 1. The choice of the lower limit ($\mathbf{H}=0$) seems arbitrary at first sight, but will be justified by the considerations which follow. We therefore obtain for the whole energy of the magnetic field the expression

$$U_m = \frac{1}{4\pi}\int dV \int_{\mathbf{H}=0}^{\mathbf{B}} \mathbf{H}\, d\mathbf{B}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which puts in evidence the fact that, for every separate element of volume, the magnetization curve is to be used in the calculation, as well as the final value of **H** or **B**.

From the expression (3) for U_m we shall first calculate the mutual force between currents and magnets. For this purpose we shall consider the most general change which U can undergo. We take an element of volume dV fixed in space. When motion of the material takes place, not only will the values of **B** and **H** in dV change, but so also will the magnetization curve corresponding to dV, so that e.g. the shaded area *abc* in fig. 1 is replaced by the area *a′b′c′*, where the contour *a′b′* is indicated by a dotted line. The most general infinitesimal change of $\int \mathbf{H}\, d\mathbf{B}$ is composed of the two strips *cbb′c′* and *abb′a′*. We have therefore

$$\delta \int \mathbf{H}\, d\mathbf{B} = \mathbf{H}\, \delta\mathbf{B} - \int_{\mathbf{H}=0}^{\mathbf{H}} \delta_H \mathbf{B}\, d\mathbf{H}.$$

We shall assume for brevity that the portion of the magnetization curve with which we have to deal is rectilinear, so that **B** has the form

$$\mathbf{B} = \mu' \mathbf{H} + 4\pi \mathbf{I}_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The residual magnetization $\mathbf{I}_0$ and the coefficient μ' are any given functions of the material and therefore of position. (It may be remarked in passing that it would lead to no essential difficulty though

we considered μ' also as a function of $\mathbf{H}$, but we shall not do so here.) From (4) we have

$$\delta_H \mathbf{B} = \mathbf{H}\,\delta\mu' + 4\pi\,\delta\mathbf{I}_0,$$

or

$$\int_0^{\mathbf{H}} \left(\frac{\partial \mathbf{B}}{\partial t}\right)_H d\mathbf{H} = \tfrac{1}{2}\frac{\partial \mu'}{\partial t}\mathbf{H}^2 + 4\pi\frac{\partial \mathbf{I}_0}{\partial t}\mathbf{H}.$$

Hence $$\frac{dU_m}{dt} = \frac{1}{4\pi}\int \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}\,dV - \frac{1}{8\pi}\int \frac{\partial \mu'}{\partial t}\mathbf{H}^2\,dV - \int \frac{\partial \mathbf{I}_0}{\partial t}\mathbf{H}\,dV. \quad (4a)$$

Suppose now that we are given the very small velocity $\mathbf{u}$, with which the individual material elements move. We have then to determine how the quantities μ' and $\mathbf{I}_0$ change with the time at the point of space considered, in consequence of this motion. In § 5, p. 91, in calculating the forces in the electric field we have admitted a possible "substantial" change in the dielectric constant K (equation (12), p. 94). We shall here neglect *magnetostriction* in chemically homogeneous media, and accordingly assume that our quantity μ' does not change for the material particle we are considering. Hence, by the equation just cited,

$$0 = \frac{\partial \mu'}{\partial t} + (\mathbf{u}\,\mathrm{grad}\,\mu').$$

We shall likewise assume that the moving particles keep their residual magnetization $\mathbf{I}_0$ unchanged. Moreover, for the sake of brevity we shall assume that those particles, for which $\mathbf{I}_0$ differs from zero, move like rigid bodies. This assumption will in practice almost always be justified. It then follows from the "substantial" constancy (p. 94) of $\mathbf{I}_0$ that the flux of $\mathbf{I}_0$ across a surface moving with the material must be constant. Thus, in equation (37*a*), p. 40, $\dot{\underline{\mathbf{I}}}_0 = 0$, or

$$\frac{\partial \mathbf{I}_0}{\partial t} = \mathrm{curl}\,[\mathbf{u}\mathbf{I}_0] - \mathbf{u}\,\mathrm{div}\,\mathbf{I}_0.$$

With the values thus found for $\partial\mu'/dt$ and $\partial\mathbf{I}_0/\partial t$, (4*a*) becomes

$$\frac{dU_m}{dt} = \frac{1}{4\pi}\int \mathbf{H}\dot{\mathbf{B}}\,dV + \frac{1}{8\pi}\int (\mathbf{u}\,\mathrm{grad}\,\mu')\mathbf{H}^2\,dV$$
$$- \int \mathbf{H}\,\mathrm{curl}\,[\mathbf{u}\mathbf{I}_0]\,dV + \int \mathbf{H}\mathbf{u}\,\mathrm{div}\,\mathbf{I}_0\,dV.$$

On the right-hand side the first and third terms must be transformed further. For the *third* we obtain by (35), p. 36, and integration over the entire system,

$$\int \mathbf{H}\,\mathrm{curl}\,[\mathbf{u}\mathbf{I}_0]\,dV = \int [\mathbf{u}\mathbf{I}_0]\,\mathrm{curl}\,\mathbf{H}\,dV$$
$$= \frac{4\pi}{c}\int [\mathbf{u}\mathbf{I}_0]\mathbf{i}\,dV = -\frac{4\pi}{c}\int \mathbf{u}[\mathbf{i}\mathbf{I}_0]\,dV.$$

To evaluate the *first* term in the expression for dU_m/dt we go back to the field equations

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c}\mathbf{i},$$

$$-\operatorname{curl} \mathbf{E} = \frac{1}{c}\dot{\underline{\mathbf{B}}},$$

multiply the first by $\mathbf{E}\,dV$, the second by $\mathbf{H}\,dV$, add and integrate over the whole system. By (35), p. 36, the integral on the left can be transformed into a surface integral, which vanishes. We are left with

$$\int \mathbf{iE}\,dV = -\frac{1}{4\pi}\int \mathbf{H}\dot{\underline{\mathbf{B}}}\,dV.$$

But, for matter moving with velocity $\mathbf{u}$, p. 40,

$$\dot{\underline{\mathbf{B}}} = \frac{\partial \mathbf{B}}{\partial t} - \operatorname{curl}[\mathbf{uB}],$$

so that we obtain

$$\frac{1}{4\pi}\int \mathbf{H}\frac{\partial \mathbf{B}}{\partial t}\,dV = -\int \mathbf{iE}\,dV + \frac{1}{4\pi}\int \mathbf{H}\operatorname{curl}[\mathbf{uB}]\,dV.$$

By (35), p. 36, and (1*a*), p. 126, the second term on the right becomes

$$\frac{1}{4\pi}\int [\mathbf{uB}]\operatorname{curl}\mathbf{H}\,dV = \frac{1}{c}\int [\mathbf{uB}]\mathbf{i}\,dV = -\frac{1}{c}\int \mathbf{u}[\mathbf{iB}]\,dV.$$

We now collect the various terms, and denote by

$$\psi = \int \mathbf{iE}\,dV, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the "thermochemical activity" (cf. p. 145), a name which will be justified in a moment. We then find, for the rate of change of U_m,

$$-\frac{dU_m}{dt} = \psi + \int (\mathbf{u}, \mathbf{f}_m)\,dV, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where

$$\mathbf{f}_m = \frac{1}{c}[\mathbf{i}, \mathbf{B} - 4\pi\mathbf{I}_0] - \frac{1}{8\pi}\mathbf{H}^2 \operatorname{grad}\mu' + \mathbf{H}\operatorname{div}(-\mathbf{I}_0). \quad (6a)$$

Equation (6) supplies complete information on the question of the disposition of the magnetic energy in a quasi-steady field, with arbitrary relative motion of circuits and magnets. First, the thermochemical energy $\psi\,dt$ is generated at the expense of U_m. In fact, since (p. 144)

$$\mathbf{i} = \sigma(\mathbf{E} + \mathbf{E}^{(e)}),$$

we have
$$\mathbf{iE} = \frac{\mathbf{i}^2}{\sigma} - \mathbf{iE}^{(e)},$$

and $$\psi = \int \frac{\mathbf{i}^2}{\sigma} dV - \int \mathbf{i}\mathbf{E}^{(e)} dV, \quad . \quad . \quad . \quad . \quad . \quad (6b)$$

so that ψ is made up of the irreversible Joule heat $\mathbf{i}^2/\sigma$, and of the work done against the impressed fields $(-\mathbf{i}\mathbf{E}^{(e)})$, which may appear as Peltier heat or in increase of the free energy of the accumulators or cells present in the circuit. To this expenditure of energy has to be added, when the material is in motion, the work done by the field per second on this account, viz. $\int(\mathbf{u}, \mathbf{f}_m)\, dV$.

The force-density represented by (6*a*) is composed of three characteristic terms:

(*a*) On an *element of volume traversed by a current* there acts in the first place the force

$$\mathbf{f} = \frac{1}{c} [\mathbf{i}, \mu'\mathbf{H}]\, dV,$$

where, by (4), p. 148, $\mu'\mathbf{H}$ is the induction, with no residual magnetism. For an element ds of a current filament of cross-section S and current strength i, we have

$$\mathbf{i}\, dV = |\, \mathbf{i}\, |\, S\, ds = i\, d\mathbf{s},$$

so that $$\mathbf{f} = \frac{i}{c} [d\mathbf{s}, \mu'\mathbf{H}]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

When there is no residual magnetism ($\mathbf{I}_0 = 0$) at the place where the wire carrying the current is situated, we find for the *force on the element* ds *of the wire*

$$\mathbf{f} = \frac{i}{c} [d\mathbf{s}, \mathbf{B}]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

(*b*) The second term of $\mathbf{f}_m$ is exactly analogous to the electrostatic force $(-\mathbf{E}^2 \operatorname{grad} K)/8\pi$. Its effect is particularly marked at the *mutual boundary of two substances for which* μ' *has different values*, the result being a tension directed outwards with respect to the surface of the more strongly magnetizable material.

(*c*) Lastly, the third term $\mathbf{H} \operatorname{div} (-\mathbf{I}_0)$ may be formally interpreted by introducing "free magnetism" $\rho_m = -\operatorname{div} \mathbf{I}_0$, acted on by the force $\mathbf{H}\rho_m$, by analogy with the electrostatic force $\mathbf{E}\rho$. In a homogeneously polarized bar magnet, for example, ρ_m is concentrated at the two ends, this concentration corresponding to the discontinuity in the normal component of $\mathbf{I}_0$.

The Maxwell stresses in the magnetic field.—For an electrostatic field we have seen (§ 8, p. 104) that the resultant force over any region can be represented by a surface integral taken over the boundary of the region. We shall show that the expression (6*a*) for the force $\mathbf{f}_m$ admits of an exactly analogous transformation. We shall in fact obtain in

this way a magnetic stress tensor $\mathbf{T}_m$, which only differs from the electric $\mathbf{T}_e$ in having $\mathbf{H}$ throughout instead of $\mathbf{E}$ (cf. (18), p. 106), and the quantity μ' introduced in (4), p. 148, instead of K. In other words, we say that (6a) is the same thing as

$$\left.\begin{aligned}(f_m)_x = \frac{\partial}{\partial x}\left(\frac{\mu'}{8\pi}(H_x^2 - H_y^2 - H_z^2)\right)\\ + \frac{\partial}{\partial y}\left(\frac{\mu'}{4\pi}H_x H_y\right) + \frac{\partial}{\partial z}\left(\frac{\mu'}{4\pi}H_x H_z\right).\end{aligned}\right\} \quad . \quad (9)$$

In the first place, whatever μ' and $\mathbf{H}$ may be, it is easily verified that the right side of this equation is identical with

$$\frac{1}{4\pi} H_x \operatorname{div}(\mu'\mathbf{H}) - \frac{1}{8\pi}\mathbf{H}^2 \frac{\partial \mu'}{\partial x} - \frac{1}{4\pi}[\mu'\mathbf{H}, \operatorname{curl}\mathbf{H}]_x. \qquad (9a)$$

But the latter expression agrees exactly with (6a), since we have

$$\operatorname{div}(\mu'\mathbf{H}) + 4\pi \operatorname{div}\mathbf{I}_0 = 0,$$
$$\operatorname{curl}\mathbf{H} = 4\pi\mathbf{i}/c.$$

The description already given (§ 8, p. 104) of the stresses in the electric field can therefore be extended word for word to the magnetic field: the force which acts upon the part of the system within any given finite region is equivalent to a system of surface forces, or

$$\int \mathbf{f}_m \, dV = \int \mathbf{T}_m \, dS.$$

The absolute value of $\mathbf{T}_m$ is

$$|\mathbf{T}_m| = \frac{\mu'}{8\pi}\mathbf{H}^2;$$

and $\mathbf{T}_m$ is in such a direction that the angle it forms with the outward normal $\mathbf{n}$ is bisected by the line of force, i.e. by the direction of $\mathbf{H}$.

3. Electric and Magnetic Units. (See also p. 251.)

In order that the units used in electromagnetism may take their place in an absolute system of measurement it is necessary, as we have seen in § 1, p. 1, to have equations connecting them with units already defined. The most obvious course is to use the expressions for the energy density of the electric and the magnetic fields, or the expressions—which for the sake of greater clearness we shall prefer—giving the electrostatic and magnetostatic forces in specially simple cases, say the Coulomb laws for the force between two electric point charges or two magnetic poles. These relations involve either electric quantities only, or else magnetic quantities only.

Whatever units be chosen, the Coulomb laws are

$$F_e = \frac{a}{K} \frac{e_1 e_2}{r^2},$$

$$F_m = \frac{b}{\mu} \frac{m_1 m_2}{r^2},$$

where e_1, e_2 are the electric charges, and m_1, m_2 the magnetic pole strengths.

The dielectric constant K and the permeability μ will be assumed here to be pure numbers without dimensions, having in a vacuum the value 1.

The numerical values and dimensions of the factors of proportionality a and b depend on the values and dimensions of the units in which e and m are measured; these units being still at our disposal, it is simplest to define them in such a way that both a and b are dimensionless and equal to 1. The unit of e, or of m, is therefore that electric charge, or that magnetic pole, which in a vacuum ($K = 1$, $\mu = 1$) repels an equal charge or pole at a distance $r = 1$ cm. with a force $\mathbf{F} = 1$ dyne. For the dimensions of the two units we have therefore

$$[e] = [m] = [M^{\frac{1}{2}} L^{\frac{3}{2}} T^{-1}].$$

The units of $[e]$ and $[m]$ being thus defined, the relations obtained in the preceding chapters enable us to determine at once the units of potential, field strength, displacement, &c. *The system of units set up in this way—called the Gaussian system—is the one which we employ in this book.*

Thus, by assuming the factors of proportionality in Coulomb's two laws to be dimensionless and equal to 1, we can define units of electric charge and magnetic pole strength, and then give absolute definitions of the units of all electric and magnetic quantities, starting from the unit of electric charge for the electric quantities, and from the unit of magnetic pole strength for the magnetic quantities. So long as we confine ourselves to electrostatics and magnetostatics, we find no connexion between the two domains; electrostatic and magnetostatic fields can exist simultaneously in the same region of space, without affecting each other at all. When we pass to electrodynamics, however, Maxwell's equations bridge the gap between the electric and the magnetic quantities. The equations are

$$c \operatorname{curl} \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + 4\pi \mathbf{i},$$

$$c \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

or, in integral form,

$$c \oint \mathbf{H}\, ds = \frac{\partial}{\partial t}\int\int D_n\, dS + 4\pi\int\int i_n\, dS,$$

$$c \oint \mathbf{E}\, ds = -\frac{\partial}{\partial t}\int\int B_n\, dS.$$

In the Gaussian system of measurement, in particular, we have the constitutive equations

$$\mathbf{D} = K\mathbf{E},$$

$$\mathbf{B} = \mu\mathbf{H},$$

with K and μ dimensionless; $\mathbf{D}$ and $\mathbf{E}$ are therefore of the same dimensions, and, in a vacuum, identical; and the same is true of the other pair, $\mathbf{B}$ and $\mathbf{H}$.

In the Gaussian system, starting from Coulomb's laws, we have now fixed the units of $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$; the factor of proportionality c, which, as experiment shows, is the same in both of Maxwell's (vector) equations, is therefore not at our disposal, but must be determined by experiment. If, for example, we send a steady current $i = \iint i_n\, dS$ round a wire ring, we can measure the magnetic field excited by the current i, and so determine the line integral $\oint \mathbf{H}\, ds$ of the magnetic field strength round a path linked with the ring; knowing i and $\oint \mathbf{H}\, ds$, we get c from the first of the fundamental equations. Or, again, we may cause magnetic flux $\iint B_n\, dS = \Phi$ to pass through the area enclosed by the ring; if we change the flux, an electromotive force $\oint \mathbf{E}\, ds$ is induced in the ring, the amount of which can be determined e.g. by an electrostatic voltmeter. The second fundamental equation then gives c. Since, as we have already mentioned, $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$ are all of the same dimensions, it follows that the dimensions of c are determined solely by the differentiations with respect to time, and length (in curl), which occur in the two fundamental equations; we see at once that its dimensions are $[LT^{-1}]$. We call c the "critical velocity"; in next chapter we shall see that electromagnetic actions are propagated with this velocity in free space. As to its numerical value, experiments of the kind just described show that

$$c = 300{,}000 \text{ km./sec.} = 3 \times 10^{10} \text{ cm./sec.}$$

Instead of proceeding simultaneously and symmetrically from the electric and from the magnetic side, as in the Gaussian system, we can follow a different method of using the connexion between the electric and magnetic quantities, as expressed in Maxwell's equations; thus we may first fix *only* the electric units, or *only* the magnetic units, exactly in fact in either case as in the Gaussian system; but then go

on to fix the other quantities by taking the constant c in Maxwell's equations to be dimensionless and equal to 1.

The so-called *electromagnetic system* e.g. is obtained as follows. We put the factor of proportionality b in the magnetic Coulomb's law equal to 1; we then find, for magnetic pole strength and the rest of the magnetic quantities, exactly the same units as in the Gaussian system. The electrical units, on the other hand, are found by putting $c = 1$ in Maxwell's equations; so that if we again consider a wire ring with magnetic flux passing through it, the electromagnetic unit of electromotive force is the E.M.F. induced in the ring when the rate of decrease of the flux, $-\partial\Phi/\partial t$, is equal to 1. If we denote the electric field strength in the Gaussian system by $\mathbf{E}$, the same field strength in electromagnetic units by $\mathbf{E}'$, then, since the right-hand (magnetic) side of Maxwell's second equation is the same ($\mathbf{H} = \mathbf{H}'$) in both systems, we have

$$c \operatorname{curl} \mathbf{E} = \operatorname{curl} \mathbf{E}',$$

and therefore

$$c\mathbf{E} = \mathbf{E}'.$$

The number measuring the electric field strength electromagnetically is accordingly c times the Gaussian number; the electromagnetic *unit* of electric field strength is therefore c times *smaller* than the Gaussian unit.

If the product of field strength and charge is to give the same force in both systems, then, the number measuring the field strength being c times greater in the electromagnetic system, the number measuring the charge must be c times smaller:

$$e' = \frac{e}{c}.$$

It then follows from the relation

$$\iint D_n \, dS = 4\pi e$$

that we must have

$$\mathbf{D}' = \frac{\mathbf{D}}{c}.$$

When we put $\mathbf{D} = c\mathbf{D}'$ in the first of Maxwell's equations, the left side of which has not changed, the factor c cancels out; in this case also the factor of proportionality c is equal to 1. It can be seen that the factors in Maxwell's equations always remain equal to each other when we change to any other system of measurement; $\mathbf{D}$, it is true, is transformed in the opposite way from $\mathbf{E}$, but it is also oppositely placed in its equation, relatively to c.

On the other hand, the relation between $\mathbf{D}$ and $\mathbf{E}$ is changed; in fact, from $\mathbf{D} = K\mathbf{E}$, with $\mathbf{E}' = c\mathbf{E}$ and $\mathbf{D}' = \mathbf{D}/c$, we find

$$\mathbf{D}' = \frac{K}{c^2}\mathbf{E}'.$$

As the result is often put: K/c^2 is the "dielectric constant" K' (which is therefore not dimensionless this time) in the electromagnetic system of units; the relation

$$\mathbf{D}' = K'\mathbf{E}'$$

is accordingly formally preserved; but K' has not, of course, the same physical reality as K.

If we begin exactly in the same way as above on the electric side, and then define the magnetic units by putting the Maxwellian factor equal to 1, we obtain the *electrostatic system* of measurement. Its electric side is of course identical with the electric side of the Gaussian system. It is hardly ever used; when electrostatic units are referred to, what is meant is the electric side of the Gaussian system.

Most of the units of the systems which have been mentioned are inconveniently large or small for measuring the quantities which occur in practice; we therefore multiply them by a suitable numerical factor, and thus obtain the *practical* system of measurement. Its importance arises from the fact that nearly all numerical data in the literature, as well as the graduation of most instruments, refer to the units of this system; to save the necessity of converting such data individually, it is often convenient to work with formulæ which have been adapted to the practical system of units.

The practical system is derived from the Gaussian system as follows:

(1) for the unit of electrical energy we take, instead of the erg, 10^7 ergs $= 1$ joule;

(2) for the unit of pressure we take 1/300 of the Gaussian unit, calling this new unit 1 volt.

We shall denote quantities expressed in practical units by a suffix 1; we have therefore

$$\mathbf{E} = \frac{1}{300}\,\mathbf{E}_1.$$

Like the unit of energy or work, the unit of force is increased 10^7 times, or

$$\mathbf{F} = 10^7\mathbf{F}_1;$$

also, since $\mathbf{F} = e\mathbf{E}$, we have

$$10^7\mathbf{F}_1 = \frac{1}{300}\,\mathbf{E}_1 e.$$

If we are to have $\mathbf{F}_1 = e_1\mathbf{E}_1$, then

$$e = 3 \times 10^9\, e_1;$$

i.e. the practical unit of charge is 3×10^9 times the Gaussian unit, and therefore $\frac{1}{10}$ of the electromagnetic unit. It is called the *coulomb*; and the unit of current (1 coulomb per second) is called the *ampere*.

It is connected with the Gaussian unit of current by the equation, similar to the one for charges,

$$i = 3 \times 10^9 i_1.$$

Since on the magnetic side no change is made from the Gaussian system, **H** and **B** remain unaltered; the unit for these is called the *gauss*; the unit of magnetic flux (1 gauss per sq. cm.) is called the *maxwell*.

We shall now write down Maxwell's equations in terms of practical units; we select the integral form, which is the one most frequently employed. Faraday's law of induction

$$c \oint \mathbf{E}\, ds = -\frac{\partial}{\partial t} \iint B_n\, dS = -\frac{\partial \Phi}{\partial t}$$

becomes
$$3 \times 10^{10} \times \tfrac{1}{300} \oint \mathbf{E}_1\, ds = -\frac{\partial \Phi_1}{\partial t},$$

or
$$\oint \mathbf{E}_1\, ds = -\frac{\partial \Phi_1}{\partial t} \times 10^{-8}:$$

the induced pressure in volts is equal to 10^{-8} *of the rate of decrease of the magnetic flux in maxwell/sec.*

The first fundamental equation, for quasi-steady currents (neglecting the displacement current), may be written

$$c \oint \mathbf{H}\, ds = 4\pi \iint i_n\, dS = 4\pi i,$$

where i is the total current threading the closed path of integration. With $i = 3 \times 10^9 i_1$, this gives

$$3 \times 10^{10} \oint \mathbf{H}_1\, ds = 4\pi \times 3 \times 10^9 i_1,$$

or
$$\oint \mathbf{H}_1\, ds = \frac{4\pi}{10} i_1:$$

the integral of the magnetic field strength round any closed path is equal to $4\pi/10$ *into the current (in amperes) threading the path.*

We have still to define the practical units of resistance, inductance, and capacity.

If we put $E = \frac{1}{300} E_1$, and $i = 3 \times 10^9 i_1$, in Ohm's law $E = iR$, we get

$$\tfrac{1}{300} E_1 = 3 \times 10^9 i_1 R,$$

so that
$$R_1 = 9 \times 10^{11} R.$$

The practical unit of resistance, called the *ohm*, is therefore $1/(9 \times 10^{11})$ of the Gaussian unit.

Inductance (L) and capacity (C) are defined by the equations

$$E = -L\frac{di}{dt}$$

and

$$\frac{dE}{dt} = -\frac{i}{C}.$$

Dimensionally, these equations are exactly similar to Ohm's law, so that we can write down at once

$$L_1 = 9 \times 10^{11} L,$$

and

$$C_1 = \frac{1}{9 \times 10^{11}} C.$$

The practical unit of inductance is called the *henry*, that of capacity the *farad*.

Finally, we have to determine the relation connecting **D** and **E** in the practical system of units. From

$$\mathbf{E} = \tfrac{1}{300}\mathbf{E}_1, \quad e = 3 \times 10^9 e_1, \text{ and}$$

$$\iint (D_n)_1 \, dS = 4\pi e_1,$$

it follows that

$$\tfrac{1}{300}\mathbf{E}_1 K = 3 \times 10^9 \mathbf{D}_1,$$

or

$$\mathbf{D}_1 = \frac{K}{9 \times 10^{11}} \mathbf{E}_1.$$

To the electrical engineer, of course, this definition of the practical system of units is of no more use than the statement, for example, that the metre is the forty-millionth part of the earth's circumference. The system must be securely based on the exact specification of two or three magnitudes which are specially easy to prepare. According to an international agreement, confirmed by statute in most countries, the units chosen for the purpose are the units of resistance and current. The so-called "international ohm" is the resistance of a column of mercury of 1 sq. mm. cross-section and length 106·3 cm. at 0° C.; the "international ampere" is the current which in one second deposits 1·118 mg. of silver. The "international volt" sends a current of 1 ampere through a resistance of 1 ohm.

These numbers once fixed will for reasons of convenience be adhered to permanently, just as the unit of length is not liable to change, although it is not exactly the forty-millionth part of the earth's meridian. Later measurements have shown that the international value of the unit of current is fairly exact; the international unit of resistance is about ·05 per cent too large. In the absence of an express statement to the contrary, all data in the literature, and the graduation of all instruments, refer to international units.

CHAPTER IX

The Electrodynamics of Quasi-steady Currents

1. The Theorem of Energy for a System of Linear Currents.

The theorems to be developed in this section are to a large extent contained in the general results of the discussion of the energy of the magnetic field given in § 2, p. 146. The great practical importance of the more special arrangements which we are about to consider may justify a separate exposition of their theory, on lines independent of our previous more general methods.

We consider a number of circuits, which will be distinguished by the indices 1, 2, . . ., k, . . ., n. Let $i_1, i_2, \ldots, i_n$ be the currents in the respective circuits, $R_1, \ldots, R_n$ their resistances, and $E_1^{(e)}, \ldots, E_n^{(e)}$ the applied electromotive forces acting in them (from accumulators, alternating current supply lines, thermal sources of E.M.F., &c.). There are to be no permanent magnets present. Further, the induction $\mathbf{B}$ is to be proportional everywhere to the field strength $\mathbf{H}$, or

$$\mathbf{B} = \mu\mathbf{H}.$$

The permeability μ may be any function of position, but is not to vary with $\mathbf{H}$ at a given point. We therefore exclude magnetically hard ferromagnetic bodies altogether. If there are any soft ferromagnetic bodies in the field, their magnetization must correspond to the rectilinear early portion of the curve (fig. 5a, p. 134). In these circumstances the magnetic field energy U_{m} is given by

$$U_{\mathrm{m}} = \frac{1}{8\pi}\int \mathbf{BH}\, dV.$$

Since $\mathbf{B}$ is solenoidal, we can introduce the vector potential $\mathbf{A}$, and replace the induction $\mathbf{B}$ by curl $\mathbf{A}$; then (p. 36)

$$\mathbf{BH} = \mathbf{H}\,\mathrm{curl}\,\mathbf{A} = \mathbf{A}\,\mathrm{curl}\,\mathbf{H} + \mathrm{div}\,[\mathbf{AH}].$$

Now curl $\mathbf{H} = 4\pi\mathbf{i}/c$. On integration over the whole system, the surface integral arising from the divergence term vanishes, so that we obtain

$$U_{\mathrm{m}} = \frac{1}{2c}\int (\mathbf{Ai})\, dV. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In a linear current of strength i, if S is the cross-section and ds an element of length of the conductor, we have

$$\mathbf{i}\, dV = |\mathbf{i}|\, S\, ds = i\, ds.$$

Since i has the same value at all sections of a wire carrying a current, the energy of the field in our system of n circuits is

$$U_{\mathrm{m}} = \frac{1}{2c} \sum_{k=1}^{n} i_k \oint_k \mathbf{A}\, ds. \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

But by Stokes's theorem the integral round the kth circuit becomes

$$\oint_k (\mathbf{A}\, ds) = \iint_k B_n\, dS = \Phi_k, \quad . \quad . \quad . \quad . \quad (1b)$$

where Φ_k again stands for *the flux of induction through the* k*th circuit.*

Hence, finally, for the energy of the field we have the expression

$$U_{\mathrm{m}} = \frac{1}{2c} \sum_{k=1}^{n} i_k\, \Phi_k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The energy of the magnetic field is obtained by multiplying the current strength of each circuit by the flux of induction through it, summing over all the circuits and dividing by 2c.

Along with the energy equation (2) we use as our second fundamental theorem Faraday's law of induction, which for the kth circuit runs:

$$i_k R_k - E_k^{(e)} = -\frac{1}{c}\frac{d\Phi_k}{dt}.$$

For the discussion of these energy questions, we make use of the concept of the thermochemical activity Ψ (cf. pp. 145, 150), the definition of which is

$$\Psi = \sum_{k=1}^{n} (i_k^2 R_k - i_k E_k^{(e)}) = -\frac{1}{c} \sum_{k=1}^{n} i_k \frac{d\Phi_k}{dt}. \quad . \quad . \quad . \quad (3)$$

Ψ_k is therefore the excess of the Joule heat $i_k^2 R_k$ developed per second over the activity of the applied E.M.F. $i_k E_k^{(e)}$. If $i_k E_k^{(e)}$ happens to be negative, it means that energy is communicated to the voltaic cell or accumulator in question. The accumulator, for example, is in that case being charged, and stores up energy to the amount of $(-i_k E_k^{(e)})$ in the form of free chemical energy. Thus in all cases Ψ is the energy gained per second in the form of heat or of chemical energy.

We also admit the possibility of the various circuits, or parts of them, moving relatively to one another. The instantaneous positions of the movable parts are to be defined by certain parameters $a_1, a_2, \ldots, a_s$. If, for example, a certain portion of wire is capable of displacement

parallel to the x-axis, a may be taken to be simply the x-co-ordinate of a definite point of this portion of wire. We define the force F_r corresponding to the displacement a_r by the following condition: when the parameter a_r changes to $a_r + da_r$, work $F_r da_r$ is thereby done on the portion of wire in question. If for instance a_r is a length, F_r is a force in the ordinary sense of the word; or if a_r is an angle, F_r is a couple. The motion of our system of wires is described by assigning the values of $a_1, \ldots, a_s$ as functions of the time. Since we exclude all forces except those arising from the field itself, the work done per second during this motion by the forces of the field is

$$W = \sum_{r=1}^{s} F_r \frac{da_r}{dt}. \quad \ldots \ldots \quad (4)$$

The quantity W accordingly represents the amount of energy gained per second in the form of *mechanical work*.

Any other forms of energy than those appearing in (2), (3), and (4) will be left out of account. The theorem of the conservation of energy for our closed system therefore takes the form:

$$\frac{dU_{\mathrm{m}}}{dt} + \Psi + W = 0. \quad \ldots \ldots \quad (5)$$

Thermochemical or mechanical energy can only be produced at the expense of the field energy U_{m}.

The state of our system at any moment is uniquely defined by the values $i_1, \ldots, i_n$ of the current strengths, and the values $a_1, \ldots, a_s$ of the parameters which specify the position of the system in space. We therefore consider the energy U_{m} and the flux of induction Φ_k as functions of these $n + s$ quantities,

$$\left.\begin{aligned} U_{\mathrm{m}} &= U_{\mathrm{m}}(i_1, \ldots, i_n;\ a_1, \ldots, a_s), \\ \Phi_k &= \Phi_k(i_1, \ldots, i_n;\ a_1, \ldots, a_s), \\ & \qquad (k = 1, 2, \ldots, n). \end{aligned}\right\} \quad \ldots \quad (6)$$

The symbol $\partial/\partial i_1$ will therefore denote differentiation with respect to i_1, with the rest of the i's and all the a's kept constant.

(1) *Processes in which no mechanical work is done.*—Processes in which the parameters a do not change are of this character. We may further confine ourselves to the case where only the current i_1 changes; we therefore take $i_2, \ldots, i_n$ and $a_1, \ldots, a_n$ to be constants. It follows then from (2) that

$$\begin{aligned} \frac{dU_{\mathrm{m}}}{dt} &= \frac{\partial U_{\mathrm{m}}}{\partial i_1} \frac{di_1}{dt} \\ &= \frac{1}{c}\left\{\tfrac{1}{2}\Phi_1 + \tfrac{1}{2}\sum_{k=1}^{n} i_k \frac{\partial \Phi_k}{\partial i_1}\right\} \frac{di_1}{dt}, \end{aligned}$$

and from (3) that $$\Psi = -\frac{1}{c}\sum_{k=1}^{n} i_k \frac{\partial \Phi_k}{\partial i_1}\frac{di_1}{dt}.$$

Since W is zero here, (5) gives, with these values of dU_{m}/dt and Ψ,

$$\left.\begin{aligned} \Phi_1 &= \sum_{k=1}^{n} i_k \frac{\partial \Phi_k}{\partial i_1}, \\ \text{and} \qquad \frac{\partial U_{\mathrm{m}}}{\partial i_1} &= \frac{1}{c}\Phi_1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

(2) *Processes in which mechanical work is done.*—The result (6a) enables us to make an important transformation of the expression for the time rate of increase of the energy, in the general case when the i's and a's vary in any manner. We have first

$$\frac{dU_{\mathrm{m}}}{dt} = \sum_{k=1}^{n} \frac{\partial U_{\mathrm{m}}}{\partial i_k}\frac{di_k}{dt} + \sum_{r=1}^{s}\frac{\partial U_{\mathrm{m}}}{\partial a_r}\frac{da_r}{dt},$$

and therefore, by (6a),

$$\frac{dU_{\mathrm{m}}}{dt} = \frac{1}{c}\sum_{k=1}^{n} \Phi_k \frac{di_k}{dt} + \sum_{r=1}^{s}\frac{\partial U_{\mathrm{m}}}{\partial a_r}\frac{da_r}{dt}. \quad . \quad . \quad . \quad (6b)$$

On the other hand, it follows at once from (2) that, in all cases,

$$2\frac{dU_{\mathrm{m}}}{dt} = \frac{1}{c}\Sigma\,\Phi_k\frac{di_k}{dt} + \frac{1}{c}\Sigma i_k \frac{d\Phi_k}{dt}.$$

The last two equations give, on subtraction,

$$\frac{dU_{\mathrm{m}}}{dt} = \frac{1}{c}\Sigma i_k\frac{d\Phi_k}{dt} - \Sigma\frac{\partial U_{\mathrm{m}}}{\partial a_r}\frac{da_r}{dt}.$$

This equation, taken along with (3) and (5), gives for the mechanical work W obtained per second

$$W = \sum_{r=1}^{s}\frac{\partial U_{\mathrm{m}}}{\partial a_r}\frac{da_r}{dt}.$$

The generalized force F_r corresponding to the parameter a_r is therefore, in view of (4),

$$F_r = \frac{\partial U_{\mathrm{m}}}{\partial a_r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6c)$$

If therefore the magnetic energy $\mathrm{U_m}$ *is given, as in* (6), *as a function of the current strengths* i *and the positional co-ordinates* a, *then the partial derivative of* $\mathrm{U_m}$ *with respect to* i_k *gives the flux of induction* (6a) *through the* k*th circuit, and that with respect to* a_r *gives the force corresponding to the co-ordinate* a_r (6c).

Special attention should be paid to the algebraic sign in (6*c*). In fact, in ordinary mechanics when the potential energy is given as a function of the positional co-ordinates, the forces are found, as we know, as the partial derivatives of *minus* the energy (or *minus* the potential) with respect to the corresponding co-ordinates. Hence, according to (6*c*), *minus* the magnetic energy plays the part of the potential. While in mechanics the forces act in such a direction that the potential energy is diminished by their action—work done at the expense of potential energy—our electrodynamic forces behave in the opposite way, viz. they act in such a direction that the energy of the magnetic field *increases*. A particularly clear example of this behaviour is shown if the currents are maintained at *constant strength* during the motion, say by suitable changes of the applied E.M.F.s (accumulators switched on or off). In that case $di_k/dt = 0$ for every k, and from (4), (6*b*), and (6*c*) we find simply

$$\left(\frac{dU_{\mathrm{m}}}{dt}\right)_{i\ \mathrm{const.}} = W. \quad . \quad . \quad . \quad . \quad . \quad (6d)$$

Thus the energy of the field increases by exactly the amount of the work done: *If the wires carrying currents move, with the current strengths kept constant, in such a way that mechanical work* W *is derived from them, then the energy of the field increases by the same amount* W. *The double energy gain of amount* 2W *per second is balanced by the work done by the applied E.M.F.s, by means of which the constancy of the currents is maintained.* In fact, by (5), we have

$$2W = -\Psi.$$

Also, by (3), $$-\Psi = \sum_{k=1}^{n} (i_k E_k^{(e)} - i_k^2 R_k).$$

Now $i_k E_k^{(e)}$ is the activity of the applied E.M.F.s in the kth circuit; hence $(-\Psi)$ is really the excess of the rate of working of the applied E.M.F.s, over the rate of development of Joule heat.

Considering the direction in which, as we have shown, the mechanical forces act, we see at once from the formula (2) for U_{m} that every wire carrying a current tends to move so as to embrace the greatest possible flux of induction. The quantitative value of the force we have already obtained (equation (8), p. 151). We shall return to this again after a discussion of equations (6*a*).

2. Self-induction and Mutual Induction.

We continue the consideration of the energy

$$U_{\mathrm{m}} = \frac{1}{2c} \sum_{k=1}^{n} i_k \Phi_k$$

of a system of wires carrying currents under the conditions of the preceding section, attending specially to the flux of induction

$$\Phi_1 = \int_1 B_n \, dS$$

which passes through the first circuit. In our system the vector $\mathbf{B}$ is uniquely defined at any point by the currents $i_1, \ldots, i_n$ in the various wires. The stipulation that μ is to be independent of $\mathbf{H}$ has the further consequence that the contributions of the individual currents to the resultant vector $\mathbf{B}$ are directly proportional to the respective current strengths. Accordingly, we can subdivide the flux of induction Φ_1 also into the contributions of the separate currents $i_1, i_2, \ldots$. We express this fact by putting

$$\frac{1}{c}\Phi_1 = L_{11}i_1 + L_{12}i_2 + \ldots + L_{1n}i_n. \quad . \quad . \quad . \quad (7)$$

The quantities L_{ik} so introduced have therefore this meaning, that cL_{hk} is the flux of induction which passes through the hth circuit, when the current $i_k = 1$ flows in the kth circuit and all the other currents are zero. Clearly L_{hk} depends *only on the relative position* of the hth and the kth circuits.

We call L_{hk} the *mutual inductance* of the two circuits h and k, when $h \neq k$; and L_{kk} is called the *self-inductance* of the kth circuit.

We can of course write down an expression corresponding to (7) for each of the n circuits, so that we have generally

$$\frac{1}{c}\Phi_h = \sum_{k=1}^{n} L_{hk}\, i_k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7a)$$

The expression for the energy thus becomes

$$U_{\mathrm{m}} = \tfrac{1}{2}\sum_{k=1}^{n}\sum_{h=1}^{n} L_{kh}\, i_h i_k. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7b)$$

The energy of the field is a homogeneous quadratic form in the current strengths. This result could also be deduced from the second of equations (6a), viz.

$$\frac{1}{c}\Phi_1 = \frac{\partial U_{\mathrm{m}}}{\partial i_1}.$$

For this gives
$$2U_{\mathrm{m}} = \Sigma i_k \frac{\partial U_{\mathrm{m}}}{\partial i_k}.$$

But, by a well-known theorem of Euler's, this equation is equivalent to the statement that U_{m} is a homogeneous quadratic function of the i's.

Moreover, (7*a*), combined with (6*a*), leads to a most important result with respect to the *symmetry* of the mutual inductances. For (7*a*) gives

$$L_{hk} = \frac{\partial}{\partial i_k}\left(\frac{1}{c}\Phi_h\right),$$

so that, by (6*a*),

$$L_{hk} = \frac{\partial^2 U_m}{\partial i_k \, \partial i_h}.$$

From this it follows that we have in all cases the symmetrical relation

$$L_{hk} = L_{kh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7c)$$

We shall employ this result at once, to transform the expression for the force in (6*c*), viz.

$$F_1 = \frac{\partial U_m}{\partial a_1},$$

for the case when the co-ordinate a_1 refers to the first circuit, so that, when only a_1 changes, all the wires excepting the first remain at rest. In this case only those of the L_{hk} will depend on a_1, for which either h or k is equal to 1. We find therefore from (7*b*)

$$F_1 = \tfrac{1}{2}\sum_{k=1}^{n} \frac{\partial L_{1k}}{\partial a_1} i_1 i_k + \tfrac{1}{2}\sum_{h=1}^{n} \frac{\partial L_{h1}}{\partial a_1} i_h i_1.$$

In the second term we can replace the idex of summation h by k, and the symmetry relation (7*c*) then gives

$$F_1 = i_1 \sum_{k=1}^{n} \frac{\partial L_{1k}}{\partial a_1} i_k,$$

or, by (7),

$$F_1 = \frac{1}{c} i_1 \frac{\partial \Phi}{\partial a_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The force "in the direction of" the co-ordinate a_1 *is, except for the factor* i_1/c, *equal to the rate (per unit of* a_1*) at which the flux of induction* Φ_1 *would increase, for motion in the direction* a_1 *with the current strengths kept constant.*

We shall illustrate this result by two simple applications.

(1) *Force on an element* ds *of a conductor carrying a current.*—Let the element ds be freely movable, by means of sliding contacts, in the direction $\mathbf{r}$. In a motion of ds represented by the unit vector $\mathbf{r}_0$, ds sweeps out the area $[\mathbf{r}_0 ds]$. The flux of induction crossing this is

$$(\mathbf{B}\,[\mathbf{r}_0\, ds]) = (\mathbf{r}_0\,[ds\,\mathbf{B}]).$$

The flux Φ_1 is increased by this amount. For the component, in the direction $\mathbf{r}$, of the force on the element ds, we have therefore, by (8),

$$F_r = \frac{1}{c} i_1(\mathbf{r}_0\,[ds\,\mathbf{B}]),$$

and, consequently, for the force itself,

$$\mathbf{F} = \frac{i}{c}[d\mathbf{s}\,\mathbf{B}], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a)$$

which agrees with (8), p. 151.

(2) *The couple acting on a plane circuit in a homogeneous field.*—If S is the area enclosed by the plane circuit, and α the angle between the normal to the area and the homogeneous field $\mathbf{B}_0$, then the part of the whole induction arising from $\mathbf{B}_0$ is

$$\Phi_0 = S \mid \mathbf{B}_0 \mid \cos\alpha.$$

Hence, by (8), the couple on our circuit round an axis perpendicular to the field, and tending to increase α, is

$$C_\alpha = \frac{i}{c}\frac{\partial \Phi_0}{\partial \alpha} = -\frac{iS}{c} \mid \mathbf{B}_0 \mid \sin\alpha.$$

In the absence of magnetizable materials $\mathbf{B}_0$ is identical with $\mathbf{H}_0$. Thus C_α is the same as the couple which would act on a magnetic needle of moment iS/c. This is exactly what we should expect in view of the equivalence of currents and magnetic shells (cf. p. 127).

3. Calculation of Inductance in some Special Cases.

In the two preceding sections, we have supposed that μ may vary in any way from place to place. If we now assume the permeability to have the same value all over the magnetic field, we can find a general formula for the mutual inductance of two circuits 1 and 2. By definition, $cL_{12}i_2$ is the flux of induction which the current i_2 in the circuit 2 causes to pass through the circuit 1. If we denote by ds_1 a line element of the circuit 1, and by ds_2 a similar element of the circuit 2, we have, by (1*b*), p. 160,

$$L_{12}i_2 = \frac{1}{c}\oint (\mathbf{A}\, d\mathbf{s}_1),$$

where $\mathbf{A}$ is the vector potential at ds_1 due to the current 2. For this, however, we have previously (cf. e.g. (3*a*), p. 129) found the value

$$\mathbf{A} = \frac{\mu i_2}{c}\oint \frac{d\mathbf{s}_2}{r},$$

so that we obtain
$$L_{12} = \frac{\mu}{c^2}\oint\oint \frac{(d\mathbf{s}_1\, d\mathbf{s}_2)}{r_{12}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

To calculate the mutual inductance we have therefore to take the scalar product of every line element of the one circuit by every line element of the other, divide by the distance between the two elements, integrate over both circuits, and finally multiply by μ/c^2.

The condition of symmetry $L_{12} = L_{21}$ is obviously satisfied. As an application we shall take a case which is important in practice, that of *two parallel, coaxial, circular currents.*

Let a_1, a_2 be the radii of the two circles, and z the distance between their planes. Consider two line elements ds_1 and ds_2, inclined to one another at an angle θ. Their distance is

$$r_{12} = \sqrt{(z^2 + a_1{}^2 + a_2{}^2 - 2a_1a_2 \cos\theta)}.$$

If in the double integral (9) we integrate first over the circuit 2, we have, since $ds_2 = a_2\, d\theta$,

$$ds_1 \oint \frac{\cos\theta\, ds_2}{r_{12}} = ds_1 \int_0^{2\pi} \frac{a_2 \cos\theta\, d\theta}{\sqrt{(z^2 + a_1{}^2 + a_2{}^2 - 2a_1a_2 \cos\theta)}}.$$

The integration over the first circuit now gives simply $\int ds_1 = 2\pi a_1$, so that we obtain

$$L_{12} = \frac{2\pi\mu}{c^2} \int_0^{2\pi} \frac{a_1a_2 \cos\theta\, d\theta}{\sqrt{(z^2 + a_1{}^2 + a_2{}^2 - 2a_1a_2 \cos\theta)}}.$$

To reduce the integral to the elliptic integrals given in tables, we put

$$k^2 = \frac{4a_1a_2}{(a_1 + a_2)^2 + z^2}, \quad \text{and} \quad \theta = \pi - 2\phi.$$

Then
$$\cos\theta = -\cos 2\phi = 2\sin^2\phi - 1,$$

and
$$\begin{aligned} r_{12} &= \sqrt{(z^2 + a_1{}^2 + a_2{}^2 - 4a_1a_2 \sin^2\phi + 2a_1a_2)} \\ &= \sqrt{\{z^2 + (a_1 + a_2)^2\}}\,\sqrt{(1 - k^2 \sin^2\phi)}. \end{aligned}$$

We thus find

$$L_{12} = \frac{4\pi\mu}{c^2} k\sqrt{(a_1a_2)} \int_0^{\frac{\pi}{2}} \frac{2\sin^2\phi - 1}{\sqrt{(1 - k^2 \sin^2\phi)}}\, d\phi.$$

But we have identically

$$\frac{2\sin^2\phi - 1}{\sqrt{(1 - k^2 \sin^2\phi)}} = \frac{1}{k^2}\left\{\frac{2 - k^2}{\sqrt{(1 - k^2 \sin^2\phi)}} - 2\sqrt{(1 - k^2 \sin^2\phi)}\right\}.$$

Hence
$$L_{12} = \frac{4\pi\mu}{c^2} \sqrt{a_1a_2} \left\{\left(\frac{2}{k} - k\right) K - \frac{2}{k} E\right\}, \qquad (10)$$

where

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2\phi)}}; \quad E = \int_0^{\frac{\pi}{2}} \sqrt{(1 - k^2 \sin^2\phi)}\, d\phi; \quad (10a)$$

so that K and E are the complete elliptic integrals of the first and second kind respectively, whose values for a given k can be obtained from tables.

We shall also work out an approximation to the value of (10), for the case when the *smallest distance* b *between the two circles is small compared to their diameters.* We therefore take $|a_1 - a_2|/a_1$ and z/a_1 both small.

Then k is nearly equal to 1, and we find for E the approximate value $\int_0^{\frac{\pi}{2}} \cos\phi\, d\phi = 1$; K, however, becomes infinite for $k = 1$. To find an approximate value for K when $k \approx 1$, we first change ϕ into $\frac{\pi}{2} - \phi$, and split up K into the two parts

$$K = \int_0^{\phi_0} \frac{d\phi}{\sqrt{(1 - k^2\cos^2\phi)}} + \int_{\phi_0}^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{(1 - k^2\cos^2\phi)}},$$

where ϕ_0 is chosen so that ϕ_0^2 is very great compared with $1 - k^2$, but very small compared with 1. We can then for a first approximation replace $\cos^2\phi$ in the first term by $1 - \phi^2$, and k^2 in the second term by 1. This gives

$$K = \int_0^{\phi_0} \frac{d\phi}{\sqrt{\{(1 - k^2) + k^2\phi^2\}}} + \int_{\phi_0}^{\frac{\pi}{2}} \frac{d\phi}{\sin\phi}. \qquad (10b)$$

Now put

$$k'^2 = 1 - k^2 = \frac{(a_1 - a_2)^2 + z^2}{(a_1 + a_2)^2 + z^2} \approx \left(\frac{b}{2a}\right)^2, \qquad (10c)$$

where b is the shortest distance between the circles, and $2a$ is their mean diameter. In the first integral we can write ϕ^2 simply instead of $k^2\phi^2$. Then elementary integration gives

$$K = \log \frac{\phi_0 + \sqrt{k'^2 + \phi_0^2}}{k'} - \log \tan \frac{\phi_0}{2}.$$

Since ϕ_0^2 is very large compared with k'^2, but very small compared with 1, we may therefore take

$$K = \log \frac{2\phi_0}{k'} - \log \frac{\phi_0}{2} = \log \frac{4}{k'}$$
$$= \log \frac{8a}{b}.$$

If then we put $k = 1$ in the terms still outstanding in (10), we find

$$L_{12} = \frac{4\pi\mu}{c^2} a \left\{ \log \frac{8a}{b} - 2 \right\}. \qquad (10d)$$

This gives the mutual inductance of two coaxial circles of nearly equal radii a, the shortest distance b between the circles being small compared to a.

We shall use the result to deduce the *self-inductance of a circular wire of radius* R, *the cross-section of which is a circle of radius* r, *where* r : R *is small.* In this case it is not allowable to regard the conductor as linear, since the energy of the magnetic field would then be infinitely great for a finite current. We therefore start from the general formula (7*b*), p. 164, which for the case of a single circuit runs

$$\frac{1}{8\pi}\iiint \mathbf{BH}\,dV = \tfrac{1}{2}Li^2.$$

We divide the integration space into two parts, the first part V_0 including the whole of space outside the wire, and the other part V_1 the space occupied by the wire. The permeabilities μ_0 and μ_1 may be different. Similarly we take

$$L = L_0 + L_1,$$

i.e. we regard L as the sum of an outer and an inner self-inductance, where

$$\left.\begin{aligned} &\frac{1}{8\pi}\iiint \mathbf{BH}\,dV_0 = \tfrac{1}{2}L_0 i^2, \\ \text{and}\qquad &\frac{1}{8\pi}\iiint \mathbf{BH}\,dV_1 = \tfrac{1}{2}L_1 i^2. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

Our assumption that r/R is small enables us to use the following approximate method. In the outer region we make the calculation as if the current i were concentrated at the axis, while within the wire we assume the field to have the value which it would have if the wire were straight and of infinite length.

To calculate L_0 from (11), we replace the action of the current i by that of a magnetic shell bounded by it; the magnetic potential ϕ_m due to this shell therefore changes suddenly by $4\pi i/c$ when we pass through the shell. Hence, by Green's theorem,

$$\frac{1}{8\pi}\iiint \mathbf{BH}\,dV_0 = \frac{i}{2c}\iint B_n\,dS.$$

Now $\iint B_n\,dS$ is the flux of induction which the linear circuit i of radius R sends through the circle of radius $R - r$; so that

$$\frac{1}{c}\iint B_n\,dS = L_0 i,$$

where L_0 is the same as the quantity L_{12} given by (10*d*), provided we put R for a, and r for b; thus

$$L_0 = \frac{4\pi\mu_0}{c^2}R\left(\log\frac{8R}{r} - 2\right). \quad . \quad . \quad . \quad . \quad (11a)$$

In the *interior of the wire*, the field at a distance y from the axis is given (p. 127) by

$$|\mathbf{H}| = \frac{2i}{cr^2}y.$$

The energy per unit length is accordingly

$$\frac{\mu_1}{8\pi}\int_0^r |\mathbf{H}|^2 2\pi y\,dy = \tfrac{1}{2}i^2 \cdot \frac{\mu_1}{2c^2}.$$

The product of this by $2\pi R$, the length of the wire, is equal to $\frac{1}{2}i^2 . L_1$, by (11); therefore

$$L_1 = \frac{\pi\mu_1}{c^2}R. \quad \ldots \ldots \quad (11b)$$

The total self-inductance of our circuit is then

$$L = L_1 + L_0 = \frac{4\pi R}{c^2}\left\{\frac{\mu_1}{4} + \mu_0\left(\log\frac{8R}{r} - 2\right)\right\}. \quad . \quad (11c)$$

In particular, for non-magnetic material ($\mu_1 = \mu_0 = 1$),

$$L = \frac{4\pi R}{c^2}\left\{\log\frac{8R}{r} - \tfrac{7}{4}\right\}. \quad \ldots \quad (11d)$$

Take the numerical example $R = 5$ cm., $r = 0{\cdot}5$ mm.

Here $$\log\frac{8R}{r} = \log_e 800 = 6{\cdot}68,$$

so that $$L = \frac{20\pi}{c^2}\{0{\cdot}25\,\mu_1 + 4{\cdot}68\,\mu_0\}\left[\frac{\text{cm.}}{c^2}\right]. \quad \ldots \quad (11e)$$

For a wire of ferromagnetic material, in air, we might have $\mu_1 \approx 500$, $\mu_0 = 1$. In this case L_1/L_0 would therefore be large, and the bulk of the energy of the field would be within the wire. In contrast to this, with nonmagnetic material ($\mu_1 = \mu_0 = 1$) only about 5 per cent of the energy resides in the interior of the wire.

The above process of subdividing L into L_0 and L_1 is physically important in another case also, viz. when we are dealing with high-frequency alternating currents. In that case the current density is not uniform over the cross-section, as we assumed it to be in the above calculation of L_1. In consequence of the *skin-effect*, which will be discussed later (p. 196), the current prefers a path near the surface of the wire, and the field is likewise confined to a thin surface layer. This results in a reduction of the value of L_1, while L_0 is scarcely affected by the change. For sufficiently rapid alternations L_1 therefore becomes 0, and $L = L_0$.

An essentially simpler calculation is that of the value of L for a long coil wound on a ring. Let r be the mean radius of the ring, S its

cross-section, n the number of turns, and μ the permeability of the core. Then inside the coil we have

$$2\pi r H = \frac{4\pi}{c} ni,$$

so that

$$\frac{\mu H^2}{8\pi} = \frac{\mu}{2\pi} \frac{n^2 i^2}{c^2 r^2}.$$

Multiplying by the volume $2\pi r S$ of the field space, we obtain

$$\tfrac{1}{2} L i^2 = \frac{\mu n^2 S}{c^2 r} i^2,$$

whence

$$L = \frac{2\mu n^2 S}{c^2 r} \left[\frac{\text{cm.}}{c^2}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11f)$$

If we wish to express L in henrys in formulæ (10) *and* (11), *we must multiply the values calculated from these formulæ by* 9×10^{11}.

Similarly the formulæ of § 5, p. 60, will give the capacity in farads, if we divide by 9×10^{11}.

4. Circuit with Resistance and Self-Inductance.

We return now to the case of a number of closed conductors with ohmic resistances R_k, applied E.M.F.s $E_k^{(e)}$, and fluxes of induction Φ_k. Then the law governing the changes in the current strengths is expressed by the equation

$$i_k R_k - E_k^{(e)} = -\frac{1}{c} \frac{d\Phi_k}{dt}.$$

If the special conditions of § 1, p. 159, are satisfied, so that in particular *the permeability* μ *is everywhere a constant of the material independent of* $\mathbf{H}$, *then* Φ_k *is a linear function of the current strengths*, or, as in (7), p. 164,

$$\frac{1}{c} \Phi_k = \sum_{r=1}^{n} L_{kr} i_r.$$

Moreover, *for circuits at rest*, the inductances L_{kr} are *constant*. The law of induction therefore gives the n equations

$$i_k R_k + \sum_{r=1}^{n} L_{kr} \frac{di_r}{dt} = E_k^{(e)} \qquad (k = 1, 2, \ldots, n) \quad . \quad (12)$$

or, written out in full,

$$i_1 R_1 + L_{11} \frac{di_1}{dt} + L_{12} \frac{di_2}{dt} + \ldots + L_{1n} \frac{di_n}{dt} = E_1^{(e)},$$

$$i_2 R_2 + L_{21} \frac{di_1}{dt} + L_{22} \frac{di_2}{dt} + \ldots + L_{2n} \frac{di_n}{dt} = E_2^{(e)},$$

. . . .

These are n equations for the n derivatives di_k/dt. Hence, if the values of i_k and $E_k^{(e)}$ are given for each k at time t, the values of the i_k's are determined by (12) for time $t + dt$. The whole course of the changes of the currents in time is therefore defined by (12), provided the applied E.M.F.s $E_k^{(e)}$ are known as functions of the time.

We shall discuss a few special cases in greater detail.

Circuit without applied E.M.F.—Here we have only one equation, and in this $E^{(e)} = 0$; i.e.

$$iR + L\frac{di}{dt} = 0,$$

the general solution of which is

$$i = i_0\, e^{-t/\theta}, \qquad (\theta = L/R). \quad . \quad . \quad . \quad . \quad (13)$$

The current diminishes at such a rate that in the time $\theta = L/R$ (the "time constant" of the circuit), it falls to the eth part of its initial value.

Circuit with periodic applied E.M.F. $E^{(e)} = E_0 \cos\omega t$.—Here we have

$$iR + L\frac{di}{dt} = E_0 \cos\omega t. \quad . \quad . \quad . \quad . \quad . \quad (13a)$$

We can find a particular solution of this non-homogeneous equation by putting

$$i = i_0 \cos(\omega t - \phi). \quad . \quad . \quad . \quad . \quad . \quad (13b)$$

(13a) then becomes

$$\cos\omega t\left\{i_0 (R\cos\phi + \omega L\sin\phi) - E_0\right\} + \sin\omega t\; i_0 (R\sin\phi - \omega L\cos\phi) = 0.$$

This equation must be satisfied for all values of t; the coefficients of $\cos\omega t$ and $\sin\omega t$ must therefore both vanish. The two equations thus obtained for i_0 and ϕ give

$$\left.\begin{aligned} i_0 \sqrt{(R^2 + \omega^2 L^2)} &= E_0 \\ \tan\phi &= \omega L/R \end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad (13c)$$

The quantity $\sqrt{(R^2 + \omega^2 L^2)}$ is called the *impedance* of the circuit, and ϕ is called the *lag* of the current behind the voltage.

5. The Vector Diagram.

The calculation of the unknown quantities in alternating current work becomes shorter and easier to follow if we use complex quantities and graphical processes. As an illustration we shall take the simple problem just discussed.

Vector diagram.—We represent the alternating quantities as vectors in the complex plane. An alternating pressure $E = E_0 e^{j\omega t}$, for example,* is represented (fig. 1) by a vector of length E_0 which makes an angle ωt with the real axis. In the course of a period the extremity of this vector describes a circle about the origin. Its projection OA on the real axis gives the real value of the pressure at the moment in question.

For calculating with such vectors we have the following rules. *Addition* of the two quanties $Ae^{j\alpha}$ and $Be^{j\beta}$ is equivalent to geometrical addition (by the parallelogram law) of the corresponding vectors. One quantity $Ae^{j\alpha}$ is *multiplied* by a second quantity $Be^{j\beta}$ by turning the first vector through an angle β in the positive (i.e. counterclockwise) sense and multiplying its length by B. Multiplication by the

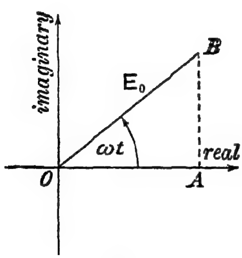

Fig. 1

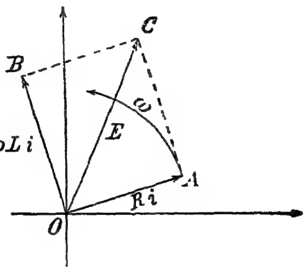

Fig. 2

imaginary unit $j = e^{j\pi/2}$ denotes rotation through $+90°$. In processes with a given value of ω—the angular velocity of the vector $Ae^{j\omega t}$—differentiation with respect to the time is equivalent to multiplication by $j\omega$, so that our alternating current equation

$$Ri + L\frac{di}{dt} = E$$

becomes the simple vector equation

$$Ri + j\omega Li = E,$$

connecting the complex vectors i and E.

For any value of i, let OA be equal to Ri (fig. 2). The vector $j\omega Li$ is then represented by OB at right angles to OA. The vector OC representing E is the vector sum of OA and OB. If we now imagine the rectangle OACB to be rotating like a rigid lamina round the origin with angular velocity ω, then the projections of OA and OC on the real axis give the values of the current and pressure at any time; and

* Here $j \equiv \sqrt{-1}$, so that $e^{j\omega t} = \cos\omega t + j\sin\omega t$. The symbol j for $\sqrt{-1}$ will be used throughout.

the imaginary axis can be used in a similar way. The results of (13c) for the impedance and the lag can be read off the figure at once.

The *activity* (rate of working) of the applied E.M.F. is at any moment

$$
\begin{aligned}
Ei &= E_0 \cos\omega t \,.\, i_0 \cos(\omega t - \phi) \\
&= \tfrac{1}{2} E_0 i_0 \{\cos\phi + \cos(2\omega t - \phi)\}.
\end{aligned}
$$

The average activity over a period is therefore

$$\overline{Ei} = \tfrac{1}{2} E_0 i_0 \cos\phi,$$

i.e. half the scalar product of the vectors E and i.

An equivalent expression for the average activity can be obtained immediately from the original real differential equation

$$iR + L\frac{di}{dt} = E^{(e)},$$

which gives $$E^{(e)} i = i^2 R + \tfrac{1}{2} L \frac{d}{dt}(i^2).$$

In time-periodic processes, such as alternating currents, the mean value of the last term is 0, so that $\overline{E^{(e)} i} = \overline{i^2 R}$; hence, on the average, the activity of the alternating current is identical with the rate of production of Joule heat.

This theorem, moreover, continues to hold in the case of the much more general equations (12), p. 171. Thus, if we multiply the typical equation of (12) by i_k, and add all the results, noting that $L_{kr} = L_{rk}$, we find, for time-periodic processes,

$$\sum_{k=1}^{n} \overline{i_k^2 R_k} = \sum_{k=1}^{n} \overline{E_k^{(e)} i_k}.$$

If we multiply (13a), p. 172, not by i, but by di/dt, we find, on taking the time average,

$$\overline{E^{(e)} \frac{di}{dt}} = L\overline{\left(\frac{di}{dt}\right)^2},$$

or, after division by ω,

$$\tfrac{1}{2} E_0 i_0 \sin\phi = \tfrac{1}{2}\omega L i_0^2.$$

Here on the right we have the peak value of the energy of the magnetic field, multiplied by ω. The quantity $\overline{E^{(e)} i} = \tfrac{1}{2} E_0 i_0 \cos\phi$ is called by electrical engineers the "*power*"; the other quantity $\tfrac{1}{2} E_0 i_0 \sin\phi$ is sometimes referred to as the "wattless" component of $\tfrac{1}{2} E_0 i_0$.

6. Two Circuits (Transformer).

We consider next two circuits in one of which there is a periodic E.M.F.—the case of a *transformer*. Here equations (12), p. 171, become (fig. 3)

$$i_1 R_1 + L_{11}\frac{di_1}{dt} + L_{12}\frac{di_2}{dt} = E^{(e)},$$

$$i_2 R_2 + L_{21}\frac{di_1}{dt} + L_{22}\frac{di_2}{dt} = 0.$$

For an alternating pressure $E^{(e)} = E_0 e^{j\omega t}$, these can be completely solved by a method analogous to the one just used, viz. we put $i_1 = i_1{}^0\, e^{j(\omega t+\phi_1)}$, and $i_2 = i_2{}^{(0)}\, e^{j(\omega t+\phi_2)}$.

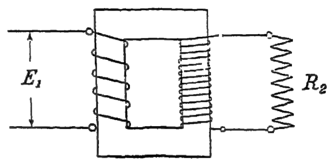

Fig. 3

We confine ourselves to a direct discussion of the relations in an ideal transformer with pure ohmic loading. This is characterized by vanishingly small resistance in the primary winding ($R_1/\omega L_{11}$ very small), and ideal fixed coupling between the primary and the secondary circuit. The latter condition means that the whole of the lines of induction which pass through the one circuit also pass through the other. This result can be attained very approximately by winding both circuits on the same closed iron core; the lines of induction will then nearly all take a course through this core. If the flux of induction in the iron core is denoted by Φ, and the number of turns in primary and secondary by n_1 and n_2, then Φn_1 and Φn_2 are the fluxes through the two windings. If $R_1 = 0$, we have therefore

$$\frac{n_1}{c}\frac{d\Phi}{dt} = E^{(e)},$$

$$i_2 R_2 + \frac{n_2}{c}\frac{d\Phi}{dt} = 0.$$

From these we have, first,

$$i_2 R_2 = -\frac{n_2}{n_1} E^{(e)}$$

for the effective pressure in the secondary. The most important point, however, is that (as we see from the first equation) the flux Φ is determined by the primary pressure $E^{(e)}$ alone, *independently of the load:*

$$j\omega\frac{n_1}{c}\Phi = E^{(e)}.$$

On the other hand, the magnetic field strength, and accordingly the flux also in the iron core, are at any one moment proportional to the sum $n_1 i_1 + n_2 i_2$:

$$n_1 i_1 + n_2 i_2 = k\Phi,$$

where k is a (real) constant of the transformer. We therefore have the vector diagram for the ideal transformer shown in fig. 4, viz. a vector E for the primary pressure, the vector $k\Phi$ lagging 90° behind this, and the vector $n_2 i_2$ with a further lag of 90°. The vector $n_1 i_1$ for the primary current is then determined as the vector difference $k\Phi - n_2 i_2$. When R_2 diminishes, i.e. when the load increases, $n_1 i_1$ revolves in the positive sense away from the direction of $k\Phi$, with a corresponding increase of the "power" (p. 174) in the primary circuit. The "wattless component" of $\frac{1}{2}E^{(c)} i_1$ (p. 174) is, on the contrary, independent of R_2.

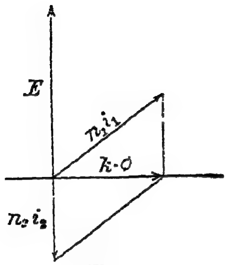

Fig. 4

7. Circuit with Self-Inductance, Capacity, and Resistance.

When a circuit includes a condenser, the current loses its solenoidal character, since the coatings of the condenser act as sources or sinks for the current. To calculate the magnetic field we should therefore in theory take into account not merely the conduction current i, but also the displacement current $\dot{\mathbf{D}}/4\pi$. So long, however, as the distance between the condenser plates is small, we can neglect the effect of the displacement current on the energy of the magnetic field, as compared with the effect of the conduction current. This simplification, as we shall see immediately, leads to a certain want of definiteness—which, however, is of no practical importance—in the application of the law of induction. Consider, for example, a circuit (fig. 5) containing a resistance R, a capacity C, and a self-inductance L, connected in series; and let the applied E.M.F. be represented by a potential difference E between the points A and F in the circuit. We calculate the pressure round the circuit along the path ARLCEFA, which runs in the wire from A to C and from C to F, but, in the condenser and from F to A, runs in the dielectric.

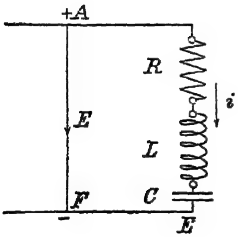

Fig. 5

The total pressure, if ϕ is the potential difference of the condenser coatings, is $Ri + \phi - E$. But the flux enclosed by the circuit clearly

depends on the course taken by the path of integration through the condenser. Thus, in taking

$$Ri + \phi - E = -L\frac{di}{dt}, \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and treating L as constant, we are ignoring that portion of the magnetic field which lies between the coatings of the condenser.

Furthermore, the current i is equal to the time rate of change of the charge on the condenser, so that, if the capacity of the condenser is C, we have

$$i = C\frac{d\phi}{dt}. \quad . \quad . \quad . \quad (14a)$$

For an alternating current varying as $e^{j\omega t}$, this gives

$$i = j\omega C\phi,$$

and therefore, from (14),

$$Ri + j\omega Li - \frac{j}{\omega C}i = E.$$

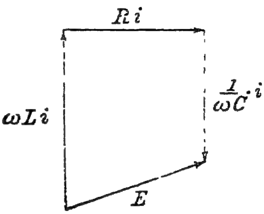

Fig. 6

It follows that i and E are related in the way represented in the vector diagram (fig. 6). The impedance of the circuit is given by

$$\left|\frac{E}{i}\right| = \sqrt{\left\{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right\}}.$$

It has a minimum value when $\omega^2 = 1/LC$. For small values of R, the minimum is very sharply defined. If then the applied pressure E is made up of partial pressures of all possible periods superimposed on each other, the current, speaking broadly, will contain only those frequencies which are close to the "proper frequency"

$$\frac{1}{2\pi\sqrt{(LC)}},$$

corresponding to $\omega = 1/\sqrt{(LC)}$. It is a matter of resonance between the period of the pressure and the "proper" period of oscillation of the circuit. The latter will now be discussed.

Electrical Proper Vibrations.—If we connect the points A and F of the circuit (fig. 5) by a thick wire (short circuit), then E becomes 0, and our equations (14) and (14a) give

$$Ri + \phi + L\frac{di}{dt} = 0,$$

$$i = C\frac{d\phi}{dt}.$$

By eliminating ϕ, we find

$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{CL}i=0. \quad . \quad . \quad . \quad . \quad (15)$$

It may be remarked that this equation, which we have deduced from the law of induction, can also be obtained very easily by a direct application of the principle of energy. For the whole field energy at any one moment is given by

$$U=U_{\mathrm{el}}+U_{\mathrm{mag}}=\tfrac{1}{2}C\phi^2+\tfrac{1}{2}Li^2.$$

Since there are no external E.M.F.s acting, the rate at which U diminishes must be equal to the rate of development of Joule heat, i.e.

$$Ri^2=-\frac{dU}{dt}=-C\phi\frac{d\phi}{dt}-Li\frac{di}{dt}. \quad . \quad . \quad (15a)$$

Since we have also $C\dfrac{d\phi}{dt}=i$, this equation transforms at once into the one given above.

The general solution of (15), with two constants of integration c_1 and c_2, is

$$i=c_1e^{k_1t}+c_2e^{k_2t}.$$

Here, k_1 and k_2 are the two quantities

$$-\frac{R}{2L}\pm\sqrt{\left(\frac{R^2}{4L^2}-\frac{1}{CL}\right)}.$$

The discharge is therefore periodic, if $\dfrac{R}{2L}<\dfrac{1}{\sqrt{(CL)}}$; non-periodic, if $\dfrac{R}{2L}>\dfrac{1}{\sqrt{(CL)}}$.

If we write for shortness

$$\delta=\frac{R}{2L},\quad \omega_0=\frac{1}{\sqrt{(LC)}}, \quad . \quad . \quad . \quad . \quad (15b)$$

then, in the periodic case, we have

$$i=Ae^{-\delta t}\sin(\sqrt{\omega_0{}^2-\delta^2}\,t+b);$$

but, for non-periodic discharge,

$$i=Ae^{-(\delta+\sqrt{\delta^2-\omega_0{}^2})t}+Be^{-(\delta-\sqrt{\delta^2-\omega_0{}^2})t}.$$

Cases in which the damping is slight are specially important in applications. These occur when δ is so small compared with ω_0 that δ^2 can be neglected in comparison with $\omega_0{}^2$. We then have

$$i=Ae^{-\delta t}\sin(\omega_0t+b).$$

The *period of a vibration* is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{(LC)}.$$

The *logarithmic decrement* D is the logarithm of the ratio of the amplitudes of two successive vibrations; or

$$D = \log \frac{e^{-\delta t}}{e^{-\delta(t+T)}} = \delta T = 2\pi \frac{\delta}{\omega_0}$$

$$= \pi R \sqrt{\left(\frac{C}{L}\right)}. \quad \ldots \ldots \ldots \quad (15c)$$

The reciprocal of D, viz. $\frac{1}{D} = \frac{\omega_0}{2\pi\delta}$,

gives the number of vibrations after which the amplitude is reduced to $1/e$ of its original value.

As a numerical example, consider a Leyden jar of radius 5 cm., thickness 0·2 cm., and height 20 cm.; by the formula for the plate condenser (p. 70),

$$C = \frac{KS}{4\pi d},$$

we find, with $K = 5$ (glass), $S = (2\pi \times 5 \times 20 + \pi \times 25)$ sq. cm.,

$$C = \frac{5 \times \pi \times 225}{4\pi \times 0{\cdot}2} = 1400 \text{ cm.}$$

$$= \frac{1400}{9 \times 10^{11}} \text{ farad.}$$

As short-circuiting wire we take a single copper wire bent to a circle, of the dimensions used in (11*e*), p. 170, viz. $D = 10$ cm., $d' = 0{\cdot}1$ cm. Its self-inductance L is, by (11*e*),

$$L = \frac{20\pi}{c^2} \times 4{\cdot}93 = 34{\cdot}4 \times 10^{-20} \frac{\text{sec.}^2}{\text{cm.}}$$

$$= 34{\cdot}4 \times 10^{-20} \times 9 \times 10^{11} \text{ henry.}$$

For the resistance R of the wire, we calculate first the ordinary resistance for direct current,

$$R = \frac{\pi \times 10 \times 4}{0{\cdot}01 \times \pi} \times 1{\cdot}7 \times 10^{-6} \text{ ohm}$$

$$= 6{\cdot}8 \times 10^{-3} \text{ ohm.}$$

With these numbers we get from (15*b*)

$$\delta = \frac{R}{2L} = 1{\cdot}1 \times 10^4 \text{ sec.}^{-1},$$

and
$$\nu_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{(LC)}} = 7{\cdot}3 \times 10^6 \text{ sec.}^{-1}.$$

This frequency would correspond to a wave-length $\lambda = c/\nu_0 = 41$ metres. The number of vibrations giving damping to the *e*th part would be $\nu_0/\delta = 660$.

If the vibrations are excited by discharge across a spark gap, we should of course expect decidedly stronger damping, on account of the increase in R due to the resistance of the spark gap. Even without a spark gap, an essentially higher value of R is to be expected, on account of the "skin effect" to be discussed in § 5, p. 196, whereby in such rapid vibrations the current is crowded towards the surface of the wire. In cases of this type, we cannot use the whole cross-section as one of the data for calculating R, but only the cross-section of that layer near the surface which is occupied by the current; the latter area is a function of the frequency.

But even after this correction our calculation of the damping still contains an inaccuracy, highly important in theory, although practically insignificant in our numerical example. In our circuit, the electrical current begins and ends at the coatings of the Leyden jar. These act as sources and sinks for the current. Nevertheless, the whole of the preceding discussion of the magnetic field is founded upon the equation for steady currents,

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c}\, i,$$

which obviously cannot be right, if div $\mathbf{i}$ is anywhere different from zero—for div curl $\mathbf{H}$ is everywhere identically 0. We have therefore, in our proof of equation (14), ignored the fact that the conduction current is interrupted by the dielectric in the condenser. A rigorous treatment would require to take into account the displacement current $\dot{\mathbf{D}}/4\pi$, by which the conduction current must be supplemented to produce the Maxwellian solenoidal total current $\mathbf{c} = \mathbf{i} + \dot{\mathbf{D}}/4\pi$. The practical effect of this term will be considered in detail later (p. 229). It corresponds to radiation of energy in the form of electromagnetic waves, radiation which can be described in terms of the Poynting vector $\mathbf{N}$ (p. 146). In the energy equation (15a), the only transformation of the field energy which we have considered is the transformation into the Joule heat Ri^2. But besides this thermal energy the action of the field is also associated with energy of radiation, the mean rate of production of which is likewise proportional to i^2, and can therefore be represented by a term $R_r i^2$. It is therefore the sum $R + R_r$ of the ohmic resistance and the resistance due to radiation which determines the damping. That we could disregard the displacement current and radiation with impunity in our treatment of the condenser discharge we owe to the facts (1) that the distance between the condenser coatings was small compared to the length of the discharging wire, and (2) that the length of the latter was itself small compared to the wave-length of the radiation corresponding to its proper frequency. It was only in consequence of these two circumstances

that we committed no very serious mistake when we treated the current in the wire as quasi-steady and so assumed that at any given moment the current across every cross-section of the wire had one and the same value. When the distance between the coatings is increased, the capacity of the condenser may become so small that the capacity of the conducting wires may become comparatively important. But in that case we have to contemplate sources and sinks of current, distributed over the whole wire, so that i will certainly have different values at different points. If, on the other hand, the length of the discharging wire is of the same order of magnitude as the wave-length, then the finite velocity of propagation leads to a relative phase displacement of the currents in the various parts of the circuit. We shall go into these matters more minutely when we come to discuss waves in wires (p. 206) and Hertzian waves (p. 223).

CHAPTER X

Electromagnetic Waves

1. Plane Waves in a Homogeneous Isotropic Dielectric.

We pass in this chapter to the discussion of electromagnetic fields which vary rapidly in space and time, especially electromagnetic waves. The assumption that the current is quasi-steady is not permissible in cases of this type. The mathematical theory of electric waves has to be based on the differential equations of the electromagnetic field for bodies at rest (§ 1, p. 143).

We consider in the first place a homogeneous isotropic dielectric; there are no applied electromotive forces, K and μ are constant, σ is equal to 0. The field equations of p. 144 become

$$\frac{K}{c}\frac{\partial \mathbf{E}}{\partial t} = \operatorname{curl} \mathbf{H}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1a)$$

$$-\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t} = \operatorname{curl} \mathbf{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1b)$$

Since μ is constant,

$$\operatorname{div} \mathbf{H} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1c)$$

Also, if there are no "true" charges (p. 74) in the interior of the insulator,

$$\operatorname{div} \mathbf{E} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1d)$$

This system of equations determines the propagation of electromagnetic waves in the dielectric. We can easily eliminate either of the vectors $\mathbf{E}$, $\mathbf{H}$ from ($1a$) and ($1b$). To eliminate $\mathbf{E}$, take the curl of the first equation; differentiate the second with respect to t, and multiply it by K/c; then add.

We thus obtain

$$-\frac{K\mu}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = \operatorname{curl}\operatorname{curl} \mathbf{H}.$$

In view of ($1c$) and the identity (33), p. 36, this becomes

$$\frac{K\mu}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = \Delta \mathbf{H}. \quad . \quad . \quad . \quad . \qquad (1e)$$

On the other hand, we may eliminate $\mathbf{H}$ instead of $\mathbf{E}$ by an exactly similar process, and so obtain

$$\frac{K\mu}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \Delta \mathbf{E}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1f)$$

The two vectors $\mathbf{E}$ and $\mathbf{H}$ accordingly satisfy the same differential equation. For fields which do not vary with the time, the equation reduces to Laplace's equation.

We shall now look for particular solutions of the field equations, corresponding to plane homogeneous wave trains. A wave train is called plane when a family of parallel planes can be taken in the field, such that the electric and magnetic field strengths are constant in magnitude and direction at all points of any given member of the family; the planes are called the wave fronts, the direction perpendicular to them the wave normal. We shall take the axis of x in the direction of the wave normal, so that the wave fronts are parallel to the plane of yz. Since $\mathbf{E}$ and $\mathbf{H}$ are to be constant in any one wave front, the partial derivatives with respect to y or z must vanish, and the field equations take the form

$$\left.\begin{array}{ll} \dfrac{K}{c}\dfrac{\partial E_x}{\partial t} = 0, & -\dfrac{\mu}{c}\dfrac{\partial H_x}{\partial t} = 0, \\ \dfrac{K}{c}\dfrac{\partial E_y}{\partial t} = -\dfrac{\partial H_z}{\partial x}, & -\dfrac{\mu}{c}\dfrac{\partial H_y}{\partial t} = -\dfrac{\partial E_z}{\partial x}, \\ \dfrac{K}{c}\dfrac{\partial E_z}{\partial t} = +\dfrac{\partial H_y}{\partial x}, & -\dfrac{\mu}{c}\dfrac{\partial H_z}{\partial t} = +\dfrac{\partial E_y}{\partial x}. \end{array}\right\} \quad . \quad . \qquad (2a)$$

$$\frac{\partial H_x}{\partial x} = 0, \quad \frac{\partial E_x}{\partial x} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2b)$$

From (2b) and the first line of (2a), it follows that the longitudinal components E_x and H_x are constant in space as well as in time. If they were different from 0, they could only represent a statical field superimposed on the wave system. Such a field, however, could have no effect on the wave propagation and is therefore of no interest for the present problem. For this reason we put

$$E_x = H_x = 0.$$

With regard to the four remaining equations of (2a) we note that two of them connect E_y and H_z, the other two E_z and H_y. We can therefore deal with these pairs of components separately. The equations

$$\frac{K}{c}\frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x}, \quad -\frac{\mu}{c}\frac{\partial H_z}{\partial t} = +\frac{\partial E_y}{\partial x} \quad . \quad . \qquad (2c)$$

give, on elimination of H_z and E_y respectively,

$$\frac{K\mu}{c^2}\frac{\partial^2 E_y}{\partial t^2}=\frac{\partial^2 E_y}{\partial x^2}, \qquad (2d)$$

$$\frac{K\mu}{c^2}\frac{\partial^2 H_z}{\partial t^2}=\frac{\partial^2 H_z}{\partial x^2}, \qquad (2e)$$

equations which might be got at once from (1e) and (1f) by deleting the derivatives with respect to y and z, which vanish for the homogeneous plane waves we are considering. The partial differential equations (2d), (2e) are familiar from their occurrence in the theory of vibrating strings. The general solution of (2d) can be written in the form

$$E_y=f_1(x-at)+f_2(x+at), \qquad (3)$$

where

$$a=\frac{c}{\sqrt{(K\mu)}}, \qquad (3a)$$

and f_1, f_2 are any functions of the single arguments $x-at$, $x+at$ respectively.

The equations (2c) are then both satisfied if we also take

$$H_z=\sqrt{(K/\mu)}\,\{f_1(x-at)-f_2(x+at)\}. \qquad (3b)$$

The arbitrary functions $f_1(x-at)$, $f_2(x+at)$ represent waves propagated in the positive and negative directions of the x-axis respectively.

We confine ourselves in what follows to the discussion of the particular solution given by the function $f_1(x-at)$. The form of the function f_1 is determined by the wave form at the time $t=0$; this wave form is propagated without change of type with the velocity a. The velocity of a plane electromagnetic wave in an isotropic insulator is therefore independent of the wave form and the wave length. In free space, where $K=1$ and $\mu=1$ in the Gaussian system of units, the velocity becomes by (3a) the universal constant c, which by comparing the electrostatic and electromagnetic units is found (p. 154) to have the value

$$c=3\times 10^{10}\ \text{cm./sec.}$$

This number agrees with the velocity of light *in vacuo*, i.e. *in free space the velocity of plane electromagnetic waves is equal to* c, *the velocity of light.*

It is not their velocity only which light waves and electromagnetic waves have in common. *Electromagnetic waves, like light waves, are transverse to the direction of propagation.* We have found, in fact, that neither $\mathbf{E}$ nor $\mathbf{H}$ can have a periodically changing longitudinal component. Both vectors are perpendicular to the wave normal. In free space, electromagnetic waves and light waves therefore behave in an exactly similar way.

These consequences of his field equations were what led Maxwell to formulate the *electromagnetic theory of light.* This theory regards light and radiant heat as phenomena of electromagnetic wave motion. It is superior to the old mechanical theory of light, first, because it enables the velocity of light to be deduced from purely electromagnetic measurements; and, secondly, because by its very nature it allows only transversal plane light waves. On the old theory, which regarded light as wave motion of an elastic medium, it was difficult to explain the non-occurrence of longitudinal light. *The electromagnetic theory excludes longitudinal light from the beginning.*

If light is really an electromagnetic process, then all the optical properties of a substance must be completely determined when its electrical constants are given. It follows in fact from the Maxwellian theory, as we shall see immediately, that the optical index of refraction is effectively determined by the dielectric constant, and the coefficient of absorption by the electric conductivity. It is only, it is true, in the case of sufficiently long waves—infra-red or Hertzian waves—that these quantitative requirements of the theory are in accord with experimental facts. The difficulty is explained and removed by the theory of electrons, which shows that electric polarization, on account of the inertia of the electrons, must be treated as a kinetic process, in which the frequency of the incident light plays a dominating part. This additional consideration—dependence of K and σ upon frequency—must be taken into account if Maxwell's theory is to give a description of optical phenomena which fits the facts.

In dielectric bodies, whose dielectric constant and magnetic permeability are different from 1, the velocity of electromagnetic waves is given by (3a). The index of refraction of a dielectric is accordingly given in the general case by

$$n = \frac{c}{a} = \sqrt{K\mu}. \quad \quad (3c)$$

In the special case when $\mu = 1$, we have the so-called "Maxwell's relation"

$$n^2 = K. \quad \quad (3d)$$

For insulators, which are neither paramagnetic nor diamagnetic, the dielectric constant, according to the electromagnetic theory of light, must be equal to the square of the optical index of refraction. By testing this result experimentally, we are in a position to judge, assuming the correctness of the electromagnetic theory of light, whether the field equations do correctly describe the behaviour of dielectrics in the presence of very rapid electrical vibrations. In point of fact, the field equations for insulators were derived from the general fundamental equations I to IV (§ 1, p. 144) by extending the relation

$$\mathbf{D} = K\mathbf{E}$$

from electrostatic fields to alternating fields of any frequency, however high. Maxwell's relation (3*d*) gives us a means of testing the legitimacy of this extension.

For many gases, as for example H_2, CO_2, N_2, O_2, the relation (3*d*) is in fairly good agreement with experiment in the visible region. Even there, however, it fails for "polar" gases (like HCl, NH_3), from the chemical behaviour of which we conjecture that they consist of ions held together in pairs by Coulomb attractions. In these gases, of course, inertia must have a much greater influence, in polarization by the alternating field of the light wave, than in the first-named gases, in which electrons only have to be moved. The Maxwell relation fails in the visible part of the spectrum in those bodies which show selective absorption in the infra-red. The failure is particularly marked e.g. in water ($K = 81$, $n = 1{\cdot}33$).

The question whether the plane of polarization of a plane-polarized ray is defined by the vector **E** or by **H** cannot be decided by means of the theory as developed up to this point. It follows, however, from the extension of the electromagnetic theory to crystals that we arrive at the correct laws of crystal optics if we refer optical aeolotropy to aeolotropy of the dielectric, and take the plane of polarization through **H** and the wave normal. It may also be mentioned, without going into details, that the laws of reflexion of light at the surface of transparent bodies, as deduced from the electromagnetic theory, are in agreement with Fresnel's formulæ, provided we take the plane of polarization of a plane-polarized ray to be at right angles to the vector **E**.

We shall now calculate the energy transmitted by the plane electromagnetic wave for which

$$E_y = f(x - at), \quad H_z = \sqrt{(K/\mu)}\, f(x - at).$$

By the energy theorems (1), (2), p. 145, the energy which passes per second across 1 sq. cm. of the yz-plane is given by the Poynting vector

$$\mathbf{N} = \frac{c}{4\pi}[\mathbf{EH}],$$

i.e.

$$N_x = \frac{c}{4\pi} E_y H_z = \frac{c}{4\pi}\sqrt{\left(\frac{K}{\mu}\right)} f^2.$$

On the other hand, the energy density u in the wave field is given (p. 145) by

$$u = \frac{1}{8\pi}(K\mathbf{E}^2 + \mu\mathbf{H}^2)$$

$$= \frac{K}{4\pi} f^2.$$

For a velocity of propagation $a = c/\sqrt{(K\mu)}$ of the plane wave we have therefore

$$N_x = au;$$

i.e. the energy which crosses 1 sq. cm. of the yz-plane per second is exactly the amount contained in a cylinder of unit cross-section and length a.

2. Plane Waves in Homogeneous Conductors.

When the homogeneous isotropic body, through which the electromagnetic wave is passing, is a conductor of electricity, equation (1a), p. 182, must be extended by including the conduction current. The general field equations of § 1, p. 144, give in this case

$$\frac{K}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\sigma\mathbf{E} = \operatorname{curl}\mathbf{H}, \quad \ldots \ldots \quad (4a)$$

$$-\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t} = \operatorname{curl}\mathbf{E}, \quad \ldots \ldots \quad (4b)$$

and, since there is no "true" magnetism,

$$\operatorname{div}\mathbf{H} = 0. \quad \ldots \ldots \ldots \quad (4c)$$

Equation (1d), p. 182, which expresses the absence of free electricity in the interior of the homogeneous dielectric, continues to hold for the waves in the interior of the homogeneous conductor. To see this, form the divergence of (4a). We find

$$\frac{K}{c}\frac{\partial}{\partial t}\operatorname{div}\mathbf{E} + \frac{4\pi\sigma}{c}\operatorname{div}\mathbf{E} = 0.$$

From this equation we have already deduced (p. 115) that the volume density of *free* electricity at every point in the field diminishes according to the law

$$\rho' = \frac{1}{4\pi}\operatorname{div}\mathbf{E} = \rho_0' e^{-t/\theta},$$

where $\theta = K/4\pi\sigma$ is the so-called "time of relaxation". The rate of decay of an original distribution of free electricity ρ_0' is therefore entirely independent of the electromagnetic disturbances which penetrate the interior of the homogeneous conductor. If, for example, we assume that at time $t = 0$ the field within the conductor was zero, then ρ_0' is also zero, and therefore ρ', the density of free electricity, is permanently zero. From the equation thus obtained, viz.

$$\operatorname{div}\mathbf{E} = 0, \quad \ldots \ldots \ldots \quad (4d)$$

and from (4a), (4b), (4c), we now conclude, exactly as in the preceding section (p. 183), that only transverse plane electromagnetic waves can be propagated in the interior of the homogeneous conductor. The elimination of E, or of H, can also be carried out in precisely the same way as at the place mentioned. We thus obtain for these two vectors the differential equations

$$\frac{K\mu}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}+\frac{4\pi\sigma\mu}{c^2}\frac{\partial \mathbf{H}}{\partial t}=\Delta \mathbf{H}, \quad . \quad . \quad . \quad . \quad (4e)$$

$$\frac{K\mu}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}+\frac{4\pi\sigma\mu}{c^2}\frac{\partial \mathbf{E}}{\partial t}=\Delta \mathbf{E}, \quad . \quad . \quad . \quad . \quad (4f)$$

which take the place of (1e), (1f), p. 182.

We again investigate homogeneous plane waves; we take the x-axis in the direction of propagation, and split up the vectors E, H, which lie in the wave fronts, into their components. As before, we consider only the components E_y, H_z, which, optically speaking, correspond to a plane wave polarized parallel to the z-axis. For E_y we have then the partial differential equation

$$\frac{K\mu}{c^2}\frac{\partial^2 E_y}{\partial t^2}+\frac{4\pi\sigma\mu}{c^2}\frac{\partial E_y}{\partial t}=\frac{\partial^2 E_y}{\partial x^2}, \quad . \quad . \quad . \quad . \quad (4g)$$

which is called the "equation of telegraphy" (cf. § 8, p. 211). The same equation has to be satisfied by H_z also.

For plane waves, with wave normals in the direction of the x-axis, we obtain from (4a), (4b)

$$\frac{K}{c}\frac{\partial E_y}{\partial t}+\frac{4\pi\sigma}{c}E_y=-\frac{\partial H_z}{\partial x},$$

$$-\frac{\mu}{c}\frac{\partial H_z}{\partial t}=\frac{\partial E_y}{\partial x}.$$

If we put $\qquad E_y=ae^{j\omega(t-px/c)}, \quad H_z=be^{j\omega(t-px/c)}, \quad . \quad . \quad . \quad (5)$

($j=\sqrt{-1}$), we find from these equations for the two amplitudes a and b, and for the index of refraction p, the equations

$$(j\omega K+4\pi\sigma)a=j\omega p b,$$

$$\mu b=pa,$$

from which we have

$$\left.\begin{aligned} p^2&=\mu K-j\,\frac{4\pi\sigma\mu}{\omega}, \\ \text{and}\quad b&=\frac{p}{\mu}\,a. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5a)$$

We therefore obtain a complex index of refraction p, i.e. *absorption of the wave,* and a complex amplitude ratio b/a; the latter denotes a *phase-displacement* of the magnetic relative to the electric vector.

In (5a) we introduce the period τ and the vacuum wave-length λ_0 of the light wave by putting

$$\tau = \frac{2\pi}{\omega}; \quad \lambda_0 = \frac{2\pi c}{\omega};$$

and we also write

$$p = n - j\kappa. \quad \ldots \ldots \quad (5b)$$

For the constants of the material thus introduced, (5a) gives the two equations

$$n^2 - \kappa^2 = K\mu,$$
$$2n\kappa = 2\mu\sigma\tau.$$

By solving these we obtain, for the "refractive index" n,

$$n^2 = \tfrac{1}{2}\mu\{\sqrt{(K^2 + 4\sigma^2\tau^2)} + K\}, \quad \ldots \quad (5c)$$

and, for the "coefficient of extinction" κ,

$$\kappa^2 = \tfrac{1}{2}\mu\{\sqrt{(K^2 + 4\sigma^2\tau^2)} - K\}. \quad \ldots \quad (5d)$$

If, finally, we write p in the form

$$p = \sqrt{(n^2 + \kappa^2)}e^{-j\gamma},$$

then the *phase angle* γ is defined by

$$\tan\gamma = \kappa/n. \quad \ldots \ldots \quad (5e)$$

In terms of the constants thus defined, equations (5) give, for the real parts of E_y and H_z,

$$\left.\begin{aligned} E_y &= ae^{-2\pi\kappa x/\lambda_0}\cos\left(\frac{2\pi t}{\tau} - \frac{2\pi n x}{\lambda_0}\right), \\ H_z &= \frac{\sqrt{(n^2 + \kappa^2)}}{\mu}ae^{-2\pi\kappa x/\lambda_0}\cos\left(\frac{2\pi t}{\tau} - \frac{2\pi n x}{\lambda_0} - \gamma\right). \end{aligned}\right\} \quad . \quad (6)$$

The coefficient κ is therefore characterized by the property that the amplitudes are reduced in the ratio $e^{2\pi}$: 1 when the wave travels a distance λ_0/κ. It follows that this distance may be regarded as a convenient measure of the range or penetrating power of the wave. The quantitative discussion of these equations is, *for metals*, greatly facilitated by the fact that in these we have usually

$$2\sigma\tau \gg K.^* \quad \ldots \ldots \quad (7)$$

For the wave-length 1μ, in the short-wave infra-red, we have e.g.

$$\tau = \lambda_0/c = 10^{-4}/(3 \times 10^{10}) = 3{\cdot}3 \times 10^{-15}.$$

Also, for copper, $\sigma = 5{\cdot}14 \times 10^{17}$, so that in this case

$$2\sigma\tau = 3400.$$

* The symbol $\gg$ means "much greater than".

With regard to the dielectric constants of metals, there is at the outset the difficulty that the customary electrostatic methods for determining them are not applicable. On the contrary, the application of equation (6) to the optical behaviour of metals is precisely the method by which we determine their dielectric constant. Absolutely unequivocal experimental results are not available, but so much may be said, that there is no ground, either experimental or theoretical, for attributing to K values of a different order of magnitude in metals from those which occur in insulators. If this is so, then in our numerical example the relation (7) is actually satisfied. Further—and this is specially important for the discussion of Hertzian waves—the relation is the better satisfied, the greater the wave-length of the waves considered.

On the other hand, for very short waves (ultra-violet and especially X-rays) the inequality (7) completely fails. In the limiting case of extremely short waves, $2\sigma\tau$ becomes even vanishingly small, so that the effect of the conductivity becomes ultimately unimportant.

In the region within which the relation (7) is valid, (5c) and (5d) become

$$n = \kappa = \sqrt{(\mu\sigma\tau)}, \quad \gamma = \tfrac{1}{4}\pi. \quad . \quad . \quad . \quad . \quad (7a)$$

For the depth of penetration, or *range*, d, of the wave we have accordingly

$$d = \frac{\lambda_0}{\kappa} = \frac{1}{\sqrt{\mu}}\sqrt{\left(\frac{\lambda_0 c}{\sigma}\right)} \text{ cm.} \quad . \quad . \quad . \quad . \quad . \quad (7b)$$

For copper ($\sigma = 5{\cdot}14 \times 10^{17}$, $\mu = 1$) we therefore obtain the following values of the range d:

λ	1 cm.	1 m.	100 m.	10 km.
d	$2{\cdot}4 \times 10^{-4}$ cm.	0·024 mm.	0·24 mm.	2·4 mm.

These are at the same time the approximate thicknesses of metal plate required to screen off the wave-lengths concerned.

For the *ratio of the electric to the magnetic field energy* (time average), we obtain from (6), (5c), and (5d), in the first place generally,

$$\frac{\overline{\mu H_z^2}}{\overline{K E_y^2}} = \frac{n^2 + \kappa^2}{\mu K} = \sqrt{\left\{1 + \left(\frac{2\sigma\tau}{K}\right)^2\right\}}. \quad . \quad . \quad (7c)$$

In an insulator ($\sigma = 0$) the energy is equally divided between the two types. In a metal, on the contrary, in the region for which (7) is valid, the energy of the field is almost entirely magnetic. Even for $\lambda = 10^{-4}$ cm., in the preceding numerical example, the electrical energy amounts to only about ·03 per cent of the total. For longer waves the proportion becomes even less.

For an experimental test of the theory the most obvious suggestion is to compare the results of experiment with the expression (7*b*), which gives the range of penetration d in terms of the conductivity. In the optical region, however, extremely thin films (thickness 10^{-6} cm. at most) would be required; these would be very difficult to prepare and employ in a manner that would be free from objection. But an alternative method, more convenient than the direct measurement of absorption, is available, viz. the measurement of reflecting power.

3. Reflecting Power of Metals.

Boundary Conditions.—In dealing with the problem of the passage of a wave from a dielectric into a metal, the simplest method of attack is to begin by determining the boundary conditions which the vectors **E** and **H** must satisfy in all circumstances, deducing these from equations (4*a*), (4*b*), p. 187, by an application of Stokes's theorem. Consider an element dS of the bounding surface between the media (1) and (2), the normal to which is in the direction of the x-axis (fig. 1). We shall calculate the line integral $\oint E_1\,ds$ for a small rectangle of breadth ξ, whose two long sides (of unit length) lie in the two media respectively and along the y-direction. We find from (4*a*) and (4*b*), p. 187,

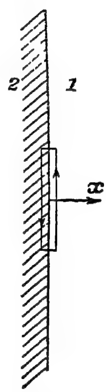

Fig. 1

$$H_{y_1} - H_{y_2} = \int_x^{x+\xi} \left(\frac{K}{c}\,\frac{\partial E_z}{\partial t} + \frac{4\pi\sigma}{c}\,E_z\right) dx,$$

$$E_{y_1} - E_{y_2} = -\int_x^{x+\xi} \frac{\mu}{c}\,\frac{\partial H_z}{\partial t}\,dx.$$

For finite values of K, σ, μ the right-hand sides vanish in the limit $\xi \rightarrow 0$, since we assume that the field strengths are in all cases finite. Hence the result: *the tangential components of* **E** *and* **H** *are continuous across the boundary between the one medium and the other.*

We now consider a plane wave, originating in free space, and incident at right angles to the surface of a metal, which we take as the plane of yz. We have therefore to put, for the complex index of refraction introduced in (5) and (5*b*), p. 188:

$$\text{for } x < 0, \quad p = 1 \qquad \text{(vacuum)},$$
$$x > 0, \quad p = n - j\kappa \qquad \text{(metal)}.$$

At the metal surface, the incident wave is split up into a reflected wave which travels in the vacuum in the negative direction of the x-axis, and a wave which penetrates into the metal (in the positive direction of the x-axis). We can satisfy Maxwell's equations for homogeneous media by the following general assumptions:

In the vacuum $x < 0$ $\begin{cases} \text{Incident wave, } E_y^{(i)} = H_z^{(i)} = ae^{j\omega(t-x/c)}, \\ \text{Reflected wave, } -E_y^{(r)} = H_z^{(r)} = a'e^{j\omega(t+x/c)}. \end{cases}$

In the metal $x > 0$ $\begin{cases} \text{Transmitted wave,} \\ \text{by (5) and (5}a\text{),} \end{cases} \begin{cases} E_y = a''e^{j\omega(t-px/c)}, \\ H_z = a''(p/\mu)e^{j\omega(t-px/c)}. \end{cases}$

The amplitudes a, a', a'' are in the first instance undefined, but the boundary conditions (at $x = 0$) lead to two relations which they must satisfy:

$$\text{continuity of } E_y\text{: } a - a' = a'',$$
$$\text{,, \quad ,, } H_z\text{: } a + a' = a''\, p/\mu.$$

If e.g. the amplitude a of the incident wave is given, we find:

$$\text{reflected wave: } a' = a\frac{p-\mu}{p+\mu},$$

$$\text{transmitted wave: } a'' = a\frac{2\mu}{p+\mu}.$$

The intensity of the radiation is in each case proportional to the square of the absolute value (modulus) of the amplitude. By the "reflecting power" R of the metal we mean the ratio of the intensities in the reflected and incident waves; if we denote the complex quantities conjugate to a and a' by $\bar{a}$ and $\bar{a}'$, we have then

$$R = \frac{a'\bar{a}'}{a\bar{a}} = \frac{(p-\mu)(\bar{p}-\mu)}{(p+\mu)(\bar{p}+\mu)}. \quad \ldots \quad (8)$$

With $p = n - i\kappa$, $\bar{p} = n + i\kappa$, this becomes

$$R = \frac{(n-\mu)^2 + \kappa^2}{(n+\mu)^2 + \kappa^2}. \quad \ldots \quad (8a)$$

In the region within which the relation (7), p. 189, is valid, we have

$$n = \kappa = \sqrt{(\mu\sigma\tau)} \gg 1.$$

In non-ferromagnetic bodies we have also $\mu = 1$, so that we obtain

$$R = \frac{2\sigma\tau - 2\sqrt{(\sigma\tau)} + 1}{2\sigma\tau + 2\sqrt{(\sigma\tau)} + 1} \approx 1 - \frac{2}{\sqrt{(\sigma\tau)}}. \quad \ldots \quad (8b)$$

This theoretical formula for the reflecting power was found by Hagen and Rubens in their experiments in the long-wave infra-red (λ *about* 25μ) *to be quantitatively verified.* They thus proved that in this long-wave region the *optical* behaviour of metals is determined by the *electrostatically* measured conductivity σ, as in equation (8*b*).

Some of their results are shown in the following table. The conductivity is expressed as the reciprocal of the resistance in ohms of

a wire 1 m. long and 1 sq. mm. in section. If the conductivity thus measured is denoted by η, we have

$$\sigma = 9 \times 10^{15}\eta.$$

The wave-lengths λ_μ are measured in microns (μ), so that λ (in cm.) $= \lambda_\mu \times 10^{-4}$. Lastly, the reflecting power is expressed as a percentage, so that $R\% = 100R$. With this notation the Hagen-Rubens formula (8*b*) becomes

$$(100 - R\%)\sqrt{\eta} = \frac{36{\cdot}5}{\sqrt{\lambda_\mu}}.$$

The quantity $(100 - R)\sqrt{\eta}$ should therefore be independent of the material, and have the value stated in the last line of the Table.

	η	$\lambda = 4\mu$		$\lambda = 8\mu$		$\lambda = 12\mu$	
		$100-R$	$(100-R)\sqrt{\eta}$	$100-R$	$(100-R)\sqrt{\eta}$	$100-R$	$(100-R)\sqrt{\eta}$
Silver	61·4	1·9	14·9	1·25	9·8	1·15	9·0
Copper	57·2	2·7	20·6	1·4	10·6	1·6	12·1
Nickel	8·5	8·2	23·9	4·65	13·6	4·1	12·0
Bismuth	0·84	24·8	22·7	18·5	16·9	17·8	16·3
$36{\cdot}5/\lambda_\mu$			19·4		13·0		11·0

For shorter waves the observed reflecting power is definitely smaller than that calculated from (8*b*). Qualitatively, this is what, on the electron theory, we ought to expect. On account of their inertia, the electrons are incapable of exactly following the rapidly changing field, and the electric current will therefore fail to attain the value $\sigma\mathbf{E}$ at each instant. In point of fact, the deviations are such as would occur if the metal possessed a smaller conductivity for the shorter waves. This inertia effect becomes even more marked in electrolytes which (like H_2SO_4 in water) statically show a high degree of conductivity, but nevertheless are perfectly transparent. Here the carriers of the current are ions with masses many thousand times greater than the masses of the electrons. It is therefore easy to understand how electrolytes should behave like insulators with respect to the electric field of the light wave.

4. The Poynting Vector in the Steady and in the Periodic Field.

In § 1, p. 145, we saw that the Poynting vector

$$\mathbf{N} = \frac{c}{4\pi}[\mathbf{EH}]$$

can be taken as representing the stream or flow of energy (per sq. cm. per sec.). For a region bounded by any closed surface, the rate of

diminution in time of the field energy contained in this region was found to be equal to the thermochemical activity, added to the flow of energy through the boundary, as given by the vector **N**. Now in a steady field the energy of the field is constant with respect to the time. Hence in this case the flow of energy, $-\int N_n\, dS$, into the region considered is equal to the thermochemical activity in the region. In a time-periodic field, on the other hand, the energy content is of course continually changing, but only in such a way that it oscillates up and down about a definite mean value. Here also, then, the mean value of the field energy taken over a period will have a constant value, so that, on a time average, the Poynting vector will again directly indicate the Joule heat developed in a closed region.

We consider first a *steady linear current maintained by applied electromotive forces*. If S is the cross-section of the conductor at any point, then in the first place, by § 3, p. 116, we have quite generally for the total current i,

$$i = \int i_n\, dS = \int \sigma(E_n + E_n^{(e)})\, dS.$$

In a "linear" current we regard σ, **E**, and $\mathbf{E}^{(e)}$ as constant over the section. Also, we can confine ourselves to the components E_a and $E_a^{(e)}$ in the direction of the axis of the wire. We denote by

$$R_0 = \frac{1}{S\sigma}$$

the ohmic resistance of the conductor per unit length, so that

$$R_0 i = E_a + E_a^{(e)}.$$

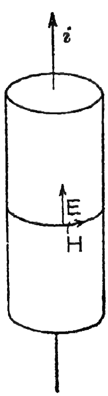

Fig. 2

We shall calculate the Poynting stream of energy through a cylindrical surface of unit length, lying in the vacuum outside but closely surrounding the wire. **E** is parallel to the axis of the wire, **H** along a circle round the axis, so that **N** is definitely perpendicular to the surface of the wire. *The flow of energy* N *into the cylinder considered* is therefore (fig. 2)

$$N = \frac{c}{4\pi}\int E_a H_c\, ds = E_a i.$$

since $\int H_c\, ds = \frac{4\pi}{c} i$. The energy flowing into the wire per unit length and unit time is therefore

$$N = R_0 i^2 - E_a^{(e)} i.$$

Hence at those places where applied E.M.F.s are acting, energy flows from the wire into the field; where Joule heat is being developed, the

energy streams back from the field into the wire. The total flow of energy, summed over the whole circuit, is of course zero, as also follows immediately—*since* $\mathbf{E}$ *is irrotational*—from the equation $N = E_a i$. When there are no E.M.F.s present, N is equal simply to the Joule heat Q developed per unit length.

In *time-periodic fields* this relation between the average values of N and Q continues to hold. We may put the result in a somewhat different form, which is convenient for many applications. We confine ourselves to oscillations which are strictly sinusoidal in respect of their variation with the time, and make use of the complex notation. The most general function $f(x, y, z, t)$ representing such an oscillation may be written

$$f = a(x, y, z) \cos\{\omega t + b(x, y, z)\}$$
$$= \text{real part of } ae^{jb}e^{j\omega t},$$

or
$$f = \text{real part of } \mathbf{a}(x, y, z)e^{j\omega t},$$

where $\mathbf{a} = ae^{jb}$ is a complex function of position only. We shall denote average values with respect to the time by a horizontal bar, and a conjugate complex value by an asterisk. Then, since $\overline{\cos\omega t} = 0$, $\overline{\cos^2\omega t} = \frac{1}{2}$, we have obviously

$$\bar{f} = 0,$$
$$\bar{f^2} = \tfrac{1}{2}a^2 = \tfrac{1}{2}\mathbf{a}\mathbf{a}^*.$$

We shall require at a later point the product of the real parts of the two complex functions

$$g = ae^{jb}e^{j\omega t}, \quad G = Ae^{jB}e^{j\omega t}.$$

If we denote the real part of g by $R_e g$, we have

$$R_e g = \tfrac{1}{2}(g + g^*), \quad R_e G = \tfrac{1}{2}(G + G^*),$$

so that
$$R_e g \,.\, R_e G = \tfrac{1}{4}(gG + g^*G^* + gG^* + g^*G).$$

But, on taking the time average, $\overline{gG} = 0$, and $\overline{g^*G^*} = 0$; also g^*G is the conjugate of gG^*. Hence

$$\overline{R_e g \,.\, R_e G} = \tfrac{1}{2}R_e(gG^*); \quad \overline{(R_e g)^2} = \tfrac{1}{2}gg^*. \quad . \quad . \quad . \quad (a)$$

The complex notation has the great advantage, in analysis relating to periodic fields, that all differentiations with respect to t in linear differential equations can be replaced by multiplication by the corresponding power of $j\omega$: e.g.

$$\frac{\partial g}{\partial t} = j\omega g; \quad \frac{\partial^2 g}{dt^2} = -\omega^2 g.$$

Maxwell's equations therefore become

$$\text{curl } \mathbf{H} = \left(\frac{j\omega K}{c} + \frac{4\pi\sigma}{c}\right)\mathbf{E}, \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

$$\text{curl } \mathbf{E} = -\frac{j\omega\mu}{c}\mathbf{H}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

In order that we may be able to apply the rule (α) to the complex vectors $\mathbf{E}$ and $\mathbf{H}$, we write (β) in the conjugate complex form (replacing j by $-j$)

$$\text{curl } \mathbf{H}^* = \left(-\frac{j\omega K}{c} + \frac{4\pi\sigma}{c}\right)\mathbf{E}^*. \quad . \quad . \quad . \quad (\beta')$$

Multiplying (β') by $c\mathbf{E}/8\pi$, and (γ) by $-c\mathbf{H}^*/8\pi$, and using (35), p. 36, we find

$$-\frac{c}{4\pi}\text{div } \tfrac{1}{2}[\mathbf{EH}^*] = \sigma \cdot \tfrac{1}{2}\mathbf{EE}^* + 2j\omega\left\{\frac{\mu}{8\pi}\tfrac{1}{2}\mathbf{HH}^* - \frac{K}{8\pi}\tfrac{1}{2}\mathbf{EE}^*\right\}. \quad (9)$$

If then we form the complex Poynting vector

$$\mathbf{N}' = \frac{c}{4\pi}\tfrac{1}{2}[\mathbf{EH}^*], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9a)$$

the real part of $-$div $\mathbf{N}'$ will give the Joule heat which is developed. For $\frac{1}{2}\mathbf{EE}^*$ is, by (α), the time-average of the square of the (real) field strength. But, besides this, the imaginary part of div $\mathbf{N}'$ is also given a concrete interpretation by (9); it represents the difference between the mean magnetic and the mean electric field energy, multiplied by 2ω. For a closed region we have therefore

$$\int(-)N_n'\,dS = Q + 2j\omega[\overline{U}_{\text{mag}} - \overline{U}_{\text{el}}], \quad . \quad . \quad (9b)$$

where Q denotes the Joule heat developed per second, and $\overline{U}$ the average field energy, magnetic or electric, within this region.

In the following section we shall apply this theorem to the interior of a wire carrying an alternating current which is not uniformly distributed over the cross-section of the wire (the skin effect). Since in this case U_{el} is always, by (7c), vanishingly small compared to U_{mag}, the real part of (9a) will give the Joule heat and therefore the ohmic resistance, while the imaginary part will give the magnetic field energy and consequently the part of the inductance contributed by the interior of the wire.

5. The Skin Effect.

We consider a straight wire of circular section and radius r_0, along which there flows an alternating current of frequency $\omega/2\pi$. At the

surface of the wire there will be an electric field in the direction of the axis, and a magnetic field at right angles to E, but also parallel to the surface. Qualitatively, therefore, the circumstances are the same as in the case considered above, that of a light wave entering the metal. We should in fact have a similar type of field near any given point of the surface if, in place of the alternating current, we had a light wave, of frequency $\omega/2\pi$, polarized perpendicular to the direction of the axis of the wire, and incident perpendicular to its surface.

For such a light wave, in the case of a metal with a plane surface, we have already (p. 190) calculated the depth the wave penetrates before the amplitude is reduced to the fraction $1/e^{2\pi}$ of its original value. For the sake of simplifying the expressions which occur, we shall employ in this section, as the measure of the depth of penetration, not precisely the quantity defined in (7*b*), p. 190, but this divided by $2\pi\sqrt{2}$, i.e. with $\mu = 1$,

$$\frac{c}{2\pi\sqrt{2}}\sqrt{\frac{\tau}{\sigma}} = \frac{c}{\sqrt{(4\pi\omega\sigma)}}.$$

Although of course the detailed analysis is rather more complicated, as we shall see immediately, for the cylindrical surface than for the plane, there are two limiting cases in which the result, qualitatively speaking, can be predicted at once.

(1) Radius of wire, r_0, small compared to $c/\sqrt{(4\pi\omega\sigma)}$. Here the alternating field has not become appreciably weaker when the wave has penetrated as far as the axis; the current density is distributed uniformly over the section of the wire.

(2) $r_0 \gg c/\sqrt{(4\pi\omega\sigma)}$. The field has completely faded away, before it has penetrated a distance which is a sensible fraction of the radius. Only in a thin superficial layer ("skin") is the current density appreciably different from 0. The whole interior of the wire is practically free from field.

It may be noted that this penetration of the alternating field into the metal follows the same law as the penetration of temperature oscillations into a body whose surface is alternately heated and cooled. In fact, since the displacement current in the metal can be neglected in comparison with the conduction current ((7), p. 189), the equation giving the oscillations in the metal, ((4*f*), p. 188), becomes

$$\frac{4\pi\sigma\mu}{c^2}\cdot\frac{\partial \mathbf{E}}{\partial t} = \Delta\mathbf{E}. \qquad \qquad (10)$$

But, for a material of specific heat γ per unit volume, thermal conductivity λ, and temperature θ, the conduction of heat is governed by the equation

$$\frac{\partial(\gamma\theta)}{\partial t} = \operatorname{div}(\lambda \operatorname{grad} \theta),$$

or
$$\frac{\gamma}{\lambda}\frac{\partial\theta}{\partial t} = \Delta\theta.$$

The intensification of the skin effect with increasing frequency is accordingly strictly analogous to the fact that the annual oscillations of temperature penetrate farther into the ground than the diurnal oscillations.

To deal with the skin effect quantitatively, we have to integrate equation (10) for the particular cross-section concerned. First, however, we shall apply the Poynting vector (9a) to the present example. We confine ourselves from the beginning to the case when the current, and hence the field also, are everywhere parallel to the axis of the wire. We take the axis of z parallel to the wire, and put $E_z = E$. By integrating (9) over unit length of the wire, and neglecting the electrical part of the field energy, we find

$$\frac{c}{8\pi}\oint EH_s^*\,ds = \sigma\int\tfrac{1}{2}EE^*\,dS + 2j\omega\,.\,\frac{\mu}{8\pi}\int\tfrac{1}{2}HH^*\,dS. \quad (11)$$

We now introduce the total current

$$i = \int\sigma E\,dS. \quad \ldots\ldots \quad (11a)$$

The time average of the square of the real current is, by (a), p. 195,

$$\overline{(R_e i)^2} = \tfrac{1}{2}ii^*.$$

The resistance R *and the "internal" self-inductance* L_i *are defined by the equations*

$$\tfrac{1}{2}Rii^* = \sigma\int\tfrac{1}{2}EE^*\,dS,$$

and
$$\tfrac{1}{2}\,.\,\tfrac{1}{2}L_i\,ii^* = \frac{\mu}{8\pi}\int\tfrac{1}{2}HH^*\,dS.$$

Thus, according to these definitions, $R\,.\,\overline{(R_e i)^2}$ is the heat developed per second, and $\tfrac{1}{2}L_i\,.\,\overline{(R_e i)^2}$ is the mean magnetic field energy in the interior of the wire. If we note further that

$$\frac{c}{4\pi}\oint H_s^*\,ds = i^*,$$

we see that (11) becomes

$$\tfrac{1}{2}(E)_S i^* = \tfrac{1}{2}Rii^* + 2j\omega\tfrac{1}{4}L_i ii^*,$$

or
$$(E)_S = Ri + j\omega L_i i, \quad \ldots\ldots \quad (12)$$

where $(E)_S$ is the value of E at the surface of the wire. *The field strength at the surface of the wire consists, therefore, of an ohmic part* Ri, *and an inductive part* ωL$_i$i, *the two parts differing in phase by* 90°.

In order to deduce the actual values of R and ωL_i from (12), we have still to express i in terms of E. Now, from (10),

$$j \cdot \frac{4\pi\omega\sigma\mu}{c^2} E = \operatorname{div} \operatorname{grad} E.$$

Integrating over the cross-section, we find, since $\int \sigma E \, dS = i$,

$$j \cdot \frac{4\pi\omega\mu}{c^2} i = \left(\frac{\partial E}{\partial r}\right)_{r=r_0} \cdot 2\pi r_0.$$

We now write

$$\left.\begin{aligned} \frac{c}{\sqrt{(4\pi\omega\sigma\mu)}} &= d \quad \text{(depth of penetration)}, \\ \frac{1}{\sigma\pi r_0^{\,2}} &= R_0 \quad \text{(resistance for direct current)}, \end{aligned}\right\} \qquad (12a)$$

and (12) becomes

$$\frac{R + j\omega L_i}{R_0} = \frac{j}{2} \frac{r_0}{d^2} \left(E \Big/ \frac{\partial E}{\partial r}\right)_{r=r_0}. \quad . \quad . \quad . \quad . \quad (13)$$

After we have integrated the differential equation (10) for E, (13) gives us at once the technically important quantities R/R_0 and ωL_i; that is to say, the increase in the resistance due to the skin effect, as well as the internal inductance.

When the section is circular, (10) becomes

$$jE = d^2 \cdot \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial E}{\partial r}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where r is the distance from the axis, and d is as defined in (12a).

The general solution of this equation involves the Bessel function of order zero, and complex argument $\sqrt{j}\, r/d$. We shall content ourselves with a direct deduction of the approximate solution in the two limiting cases mentioned above ($r_0 \gg d$ and $r_0 \ll d$).

Strong Skin Effect ($r_0 \gg d$).—In this particularly simple case the action is confined to the neighbourhood of the surface, $r \approx r_0$, so that the factor r within the bracket in (14) can be treated as constant, i.e. the surface may be regarded as plane. We thus have

$$\frac{j}{d^2} E = \frac{\partial^2 E}{\partial r^2},$$

$$E = E_0 e^{\sqrt{j} r/d}; \quad \frac{\partial E}{\partial r} = \frac{\sqrt{j}}{d} E_0 e^{\sqrt{j} r/d},$$

so that, by (13),

$$\frac{R + j\omega L_i}{R_0} = \frac{\sqrt{j}\, r_0}{2d} = \frac{r_0}{2\sqrt{2}\, d}(1 + j).$$

The ohmic resistance is therefore increased in the ratio $r_0 : 2\sqrt{2}\,d$ *as compared with its direct-current value.*

Weak Skin Effect ($r_0 \ll d$).—In (14) we take a new independent variable $y = r/d$, and put $y_0 = r_0/d$. The equation becomes

$$jE = \frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial E}{\partial y}\right).$$

We may expand the required function E in powers of y^2:

$$E = (a_0 + a_1 y^2 + a_2 y^4 + a_3 y^6 + \ldots).$$

We have therefore the identity

$$j\sum_{\nu=0}^{\infty} a_\nu y^{2\nu} = \sum_{\nu=1}^{\infty} (2\nu)^2 a_\nu y^{2\nu-2},$$

and therefore, to determine a_ν, the recurrence equation

$$a_\nu = \frac{ja_{\nu-1}}{(2\nu)^2}.$$

Hence $\quad a_1 = a_0\frac{j}{2^2};\;\; a_2 = a_0\frac{j^2}{2^2 \,.\, 4^2};\;\; \ldots;\;\; a_\nu = a_0\frac{j^\nu}{(2^\nu \nu!)^2}.$

These coefficients now give in (13)

$$\frac{R + j\omega L_i}{R_0} = \frac{j}{2}y_0\left(E\Big/\frac{\partial E}{\partial y}\right)_{y=y_0}$$

$$= \frac{j}{2}\,\frac{\sum\limits_{0}^{\infty} a_\nu y_0^{2\nu}}{\sum\limits_{1}^{\infty} 2\nu a_\nu y_0^{2\nu-2}}.$$

If we again expand on the right in powers of y_0, and put in the values found for $a_1, a_2, a_3, \ldots$, we find

$$\frac{R + j\omega L_i}{R_0} = 1 + j\left(\frac{y_0}{2\sqrt{2}}\right)^2 + \frac{1}{3}\left(\frac{y_0}{2\sqrt{2}}\right)^4 + \ldots.$$

As an index of the strength of the skin effect we may use the number z, where

$$z = \frac{r_0}{2\sqrt{2}\,d} = \frac{r_0\sqrt{(\pi\omega\sigma)}}{\sqrt{2}\,c} = \pi r_0\sqrt{\left(\frac{\sigma}{c\lambda}\right)} = \sqrt{\left(\frac{\pi}{R_0 c\lambda}\right)}. \quad (14a)$$

We then have the results:

$$\left.\begin{aligned} &\text{When } z \text{ is small: } \frac{R}{R_0} = 1 + \tfrac{1}{3}z^4 + \ldots;\;\; L_i = \frac{1}{2c^2};\\ &\text{When } z \text{ is large: } \frac{R}{R_0} = \frac{\omega L_i}{R_0} = z. \end{aligned}\right\} \quad (14b)$$

The relationship between R_0 and z is shown graphically in fig. 3. The index numbers z for copper wire of various radii, and for various wave-lengths λ, are tabulated below. For the calculation, the value of $\sqrt{(\sigma/c)}$ for copper is taken as $4{\cdot}2 \times 10^3$ (cm.)$^{-\frac{1}{2}}$. This gives, by (14*a*), $z/\pi = 4{\cdot}2 \times 10^3 r_0/\sqrt{\lambda}$.

TABLE OF VALUES OF z/π

$r_0 =$	1	0·1	0·01	Number of Periods per second $= c/\lambda$.
$\lambda = 6 \times 10^8$ cm.	0·17	0·017	0·0017	50
$\lambda = 6 \times 10^6$ cm.	1·7	0·17	0·017	5000
$\lambda = 6 \times 10^4$ cm.	17	1·7	0·17	5×10^5
$\lambda = 6$ cm.	1700	170	17	5×10^9

Intensity of the skin effect in copper wire;
r_0 = radius, λ = wave-length.

A phenomenon closely related to the skin effect occurs in the so-called high-frequency heating of cylindrical rods. The rod to be heated is placed in a longitudinal magnetic alternating field of high frequency. This field generates in the rod an electric field, the lines of force of which are circles round the axis. The Joule heat of the currents thus induced causes the increase of temperature which is desired. Here, therefore, we have again at the surface of the rod the same electromagnetic condition as would be produced by a plane polarized wave incident at right angles to the surface; only in this case the wave is polarized parallel to the axis of the wire. Comparing this case with that of the skin effect, we see that the magnetic and electric intensities have changed places with each other. In particular, equation (14) is now the equation defining the penetration of the magnetic field into the rod which is being heated. When that equation has been integrated, the complex Poynting vector gives immediately in this case also the Joule heat, and the magnetic field energy, in the rod.

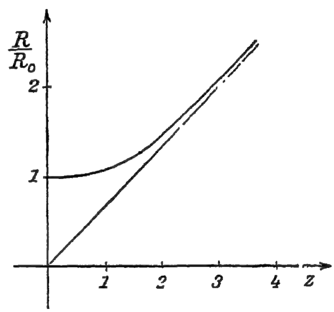

Fig. 3

6. Self Inductance and Capacity of Twin Circuits.

As a preliminary to the theory of waves in wires, we shall in this section deal with the *steady* field of a double circuit.

We consider two straight cylindrical conductors parallel to one another, the lengths of which are very great compared to their distance

apart; e.g. two parallel wires (a twin circuit in the proper sense), or a wire (a cable) enclosed in an insulating cover and laid in sea-water. In the latter case, the wire would represent the one conductor, and the sea-water the other. The two conductors carry opposite electrostatic charges, and are also traversed by currents, which are equal to each other, but in opposite directions. The resistance of the conductors will in the first instance be taken as nil, so that there is no pressure drop along the wires. It follows that both the electrical field **E** of the statical surface charge and the magnetic field **H** of the currents are everywhere perpendicular to the direction of the wires. Hence, if we take the xy plane perpendicular to that direction, we shall have $E_z = H_z = 0$.

We shall calculate the field in the insulator (K, μ), regarded as homogeneous, between the two conductors. The electric field is irrotational, and so derivable from a potential $\Phi(x, y)$:

$$E_x = -\frac{\partial \Phi}{\partial x}, \quad E_y = -\frac{\partial \Phi}{\partial y}. \quad . \quad . \quad . \quad . \quad (15)$$

The magnetic field, being solenoidal, can be derived from a vector potential. Since, however, all the currents are parallel to the z-axis, the vector potential has only the one component, viz. $A_z = \mu\Psi(x, y)$, so that

$$H_x = \frac{\partial \Psi}{\partial y}, \quad H_y = -\frac{\partial \Psi}{\partial x}. \quad . \quad . \quad . \quad . \quad (15a)$$

These two functions in the plane of the cross-section have to satisfy certain conditions, obtained as follows.

Within the insulator, which is free from charges and currents, we have div **E** $= 0$ and curl **H** $= 0$, i.e.

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0; \quad \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0. \quad . \quad . \quad . \quad (16)$$

We denote by $+e$ the charge per unit length of the conductor 1, by i the current (in the z-direction) in the same conductor. If ds is the line element of a path of integration which encircles the first conductor but not the second, then

$$4\pi e = K \oint E_n \, ds, \quad \text{and } 4\pi i/c = \oint (\mathbf{H} \, d\mathbf{s}),$$

the normal n being drawn outwards from the conductor. The element of length $d\mathbf{s}$, with the absolute value ds, and components dx and dy, is connected with the components n_x and n_y of the unit vector **n** by the equations

$$dx = -n_y \, ds, \quad dy = n_x \, ds. \quad . \quad . \quad . \quad . \quad (16a)$$

Thus $(\mathbf{H}\,ds) = H_x\,dx + H_y\,dy$

$$= -\,ds\left(\frac{\partial\Psi}{\partial x}n_x + \frac{\partial\Psi}{\partial y}n_y\right) = -\,ds\,\frac{\partial\Psi}{\partial n}.$$

Again, since $E_n = -\,\partial\Phi/\partial n$, we have the further conditions

$$\frac{4\pi e}{K} = -\int_1 \frac{\partial\Phi}{\partial n}\,ds,\quad \frac{4\pi i}{c} = -\int_1 \frac{\partial\Psi}{\partial n}\,ds; \quad . \quad . \quad (17)$$

and similarly, for every path of integration encircling the second conductor,

$$\frac{4\pi e}{K} = \int_2 \frac{\partial\Phi}{\partial n}\,ds,\quad \frac{4\pi i}{c} = \int_2 \frac{\partial\Psi}{\partial n}\,ds. \quad . \quad . \quad . \quad (17a)$$

There is an obvious analogy between the functions Φ and Ψ. To make the analogy complete, we now make the restrictive assumption that the magnetic field at the surface of the conductor is purely tangential. This assumption is certainly correct for a single-core cable having the form of a circular cylinder, as is clear from the symmetry. For a twin circuit, consisting of two parallel wires, it is *approximately* correct, provided that the distance between the wires is great compared to their diameter. (We shall have to apply our results later to rapidly varying fields. In this case, so long as the wire can be treated as a perfect conductor, the above assumption is always strictly correct; for an alternating field does not penetrate into such a conductor, and $\mathbf{H}$ is solenoidal; whence the normal component of $\mathbf{H}$ must vanish at the surface.) But the lines of magnetic force coincide with the curves $\Psi = \text{const.}$, and our assumption is therefore equivalent to this, that at the surfaces of the conductors 1 and 2, Ψ takes the constant values Ψ_1 and Ψ_2 respectively.

The corresponding equation of course always holds for the electrostatic potential Φ. Thus

$$\left.\begin{array}{l}\Psi = \text{const.}\\ \Phi = \text{const.}\end{array}\right\}\ \left.\begin{array}{c}\text{on the surfaces of the}\\ \text{conductors 1 and 2.}\end{array}\right\} \quad . \quad . \quad . \quad (18)$$

The functions Φ and Ψ are uniquely defined—except for an additive constant of no importance—by the three conditions (16), (17), (18). Further, since both functions (apart from the numerical values in (17)) have the same conditions to satisfy, they are *identical except for a numerical factor*. In fact, we see from (17) that

$$\Psi(x, y) = \frac{K}{c}\,\frac{i}{e}\,\Phi(x, y). \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The vectors $\mathbf{E}$ and $\mathbf{H}$ are everywhere at right angles to each other. Their numerical values are likewise in a constant ratio:

$$|\mathbf{H}| = |\mathbf{E}|\,.\,\frac{Ki}{ce}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19a)$$

A physical interpretation of Ψ may be derived from (17a) as follows. We consider an arbitrary curve s joining the two points a and b in the plane of xy, and define its normal by (16a). We displace this curve through 1 cm. parallel to the z-axis, and propose to calculate the flux of force $\int_a^b H_n\,ds$ which passes through the strip $abb'a'$ thus generated. Clearly, from (15a) and (16a) we have

$$H_n\,ds = H_x n_x\,ds + H_y n_y\,ds$$
$$= \frac{\partial \Psi}{\partial y}\,dy + \frac{\partial \Psi}{\partial x}\,dx = \frac{\partial \Psi}{\partial s}\,ds,$$

so that

$$\int_a^b H_n\,ds = \Psi_b - \Psi_a.$$

We now imagine a strip of the kind mentioned, of unit breadth, to be attached at its ends to the two conductors respectively; and we define the "external self-inductance" L_e per unit length of our double circuit to be the flux of induction (multiplied by $1/c$) which passes through this strip when the current strength i is equal to 1. Hence

$$\frac{\mu}{c}\int_2^1 H_n\,ds = \frac{\mu}{c}(\Psi_1 - \Psi_2)$$
$$= L_e i. \qquad \qquad (20)$$

(The name "external self-inductance" and the subscript e are intended as a reminder that a contribution L_i from the magnetic field within the conductors must be taken into account if we wish to calculate the total self-inductance L.)

Next, we define in the ordinary way the capacity C of the double circuit per unit length by the equation

$$e = C(\Phi_1 - \Phi_2). \qquad \qquad (20a)$$

But, by equation (19),

$$\frac{\Phi_1 - \Phi_2}{e} = \frac{\Psi_1 - \Psi_2}{i}\,\frac{c}{K}.$$

Hence

$$\frac{1}{C} = L_e \frac{c^2}{\mu K},$$

or

$$\frac{1}{\sqrt{(CL_e)}} = \frac{c}{\sqrt{(\mu K)}} = V. \qquad \qquad (21)$$

The reciprocal of the product of the capacity and the external self-inductance is equal to the square of the velocity V *of light in the surrounding medium.*

We arrive at the same result, if we define C and L_e in terms of the field energy stored up in the *insulator* (always per unit length of the double circuit):

$$U_{\text{mag}} = \tfrac{1}{2}L_e i^2;\; U_{\text{el}} = \frac{1}{2C}e^2. \qquad \qquad (22)$$

For it follows from (19a) that

$$\frac{U_{\mathrm{el}}}{U_{\mathrm{mag}}} = \frac{K\int \mathbf{E}^2\, dS}{\mu \int \mathbf{H}^2\, dS} = \frac{K}{\mu}\,\frac{c^2 e^2}{K^2 i^2} = \frac{c^2}{\mu K}\,\frac{e^2}{i^2}.$$

(Here again the part of the magnetic field energy, $\frac{1}{2}L_i i^2$, associated with the interior of the conductors is left out of account.) Taken along with (22), this gives (21) at once.

The Poynting vector of the system is everywhere parallel to the z-axis. We have in fact, from (15) and (15a),

$$\begin{aligned} N_z &= \frac{c}{4\pi}(E_x H_y - E_y H_x) \\ &= \frac{c}{4\pi}\left(\frac{\partial \Phi}{\partial x}\,\frac{\partial \Psi}{\partial x} + \frac{\partial \Phi}{\partial y}\,\frac{\partial \Psi}{\partial y}\right). \end{aligned}$$

In virtue of (16), Green's theorem now gives for the total stream of energy through the xy plane

$$\begin{aligned} N &= \int N_z\, dx\, dy \\ &= -\frac{c}{4\pi}\int_1 \Phi\,\frac{\partial \Psi}{\partial n}\, ds - \frac{c}{4\pi}\int_2 \Phi\,\frac{\partial \Psi}{\partial n}\, ds. \end{aligned}$$

On account of the constancy of Φ on the surfaces of the two conductors, we have from the second equations of (17) and (17a)

$$N = (\Phi_1 - \Phi_2)i.$$

We may note that this would be the value of the Joule heat, if we regarded the double circuit (which has no resistance) as having a resistance given by $R = (\Phi_1 - \Phi_2)/i$.

In conclusion, we shall find the values of C explicitly in two simple cases.

Twin Circuit in the Strict Sense.—Let the distance d between the two wires be very great compared to their radius b. Then we have, at any point in the field (cf. e.g. fig. 3, p. 130),

$$\Phi = -\frac{2e}{K}\log r_1 + \frac{2e}{K}\log r_2.$$

At the surface of 1, we have $r_1 = b$, and (since $d \gg b$) $r_2 \approx d$, so that

$$\Phi_1 = \frac{2e}{K}\log\frac{d}{b}, \quad \Phi_2 = -\frac{2e}{K}\log\frac{d}{b}.$$

Hence

$$C = \frac{e}{\Phi_1 - \Phi_2} = \frac{K}{4\log(d/b)}, \quad \ldots \ldots \quad (23a)$$

for a twin circuit, with $d \gg b$.

Concentric Cable (copper wire of radius a, insulation of radius b, beyond $r = b$ sea-water as second conductor).

In the insulator $\Phi = -(2e/K)\log r$. Hence

$$\Phi_1 = -\frac{2e}{K}\log a; \quad \Phi_2 = -\frac{2e}{K}\log b.$$

Thus, for a cable, $$C = \frac{e}{\Phi_1 - \Phi_2} = \frac{K}{2\log(b/a)}. \qquad (23b)$$

It may be noted that in the expressions (23*a*) and (23*b*) C is a pure number which, in such applications as occur in practice, will lie, say, between 1 and $\frac{1}{10}$. Similarly, the order of magnitude of L_e by (21), is from 10^{-21} to 10×10^{-21}.

7. Waves along Perfectly Conducting Wires.

We consider the same twin circuit (two parallel wires) as in the preceding section; the field is to be transverse as before, but is now *to depend on* t *as well as on* z. We first write down Maxwell's equations for the insulator, for the case in which $H_z = E_z = 0$:

$$\frac{K}{c}\dot{\mathbf{E}} = \operatorname{curl}\mathbf{H}, \text{ or } \begin{cases} \dfrac{K}{c}\dfrac{\partial E_x}{\partial t} = -\dfrac{\partial H_y}{\partial z} & (a) \\[2ex] \dfrac{K}{c}\dfrac{\partial E_y}{\partial t} = \dfrac{\partial H_x}{\partial z} & (b) \\[2ex] 0 = \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y}, & (c) \end{cases}$$

$$\frac{\mu}{c}\dot{\mathbf{H}} = -\operatorname{curl}\mathbf{E}, \text{ or } \begin{cases} \dfrac{\mu}{c}\dfrac{\partial H_x}{\partial t} = \dfrac{\partial E_y}{\partial z} & (d) \\[2ex] \dfrac{\mu}{c}\dfrac{\partial H_y}{\partial t} = -\dfrac{\partial E_x}{\partial z} & (e) \\[2ex] 0 = \dfrac{\partial E_x}{\partial y} - \dfrac{\partial E_y}{\partial x}, & (f) \end{cases} \qquad (24)$$

$$\operatorname{div}\mathbf{E} = 0, \text{ or } \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0, \qquad (g)$$

$$\operatorname{div}\mathbf{H} = 0, \text{ or } \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0. \qquad (h)$$

So far as dependence on x and y is concerned, these equations are exactly the same as in the steady case. We can therefore again introduce functions Φ and Ψ which now, however, depend on z and t as well as on x and y.

Thus the four equations (c, f, g, h) are satisfied by

$$\left.\begin{aligned} E_x = -\frac{\partial\Phi}{\partial x}, \quad H_x = \frac{\partial\Psi}{\partial y} \\ E_y = -\frac{\partial\Phi}{\partial y}, \quad H_y = -\frac{\partial\Psi}{\partial x} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (25a)$$

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0, \quad \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = 0. \quad . \quad . \quad (25b)$$

If $+e$ and $-e$ are again the charges per unit length on the two conductors for the section considered, and i and $-i$ are the current strengths, where of course e and i are now functions of z and t, then Φ and Ψ have to satisfy all the conditions (16), (17), (18) laid down in the preceding section (pp. 202, 203). The conditions (18) are in the present case—an alternating field—*rigorously* satisfied, since the field in the conductor vanishes. Next, the tangential component of **E** is continuous at the surface, likewise the normal component of **H**. (The tangential component of **H**, on the contrary, in a *perfect* conductor is in general discontinuous, on account of the "skin" current at the surface of the conductor.) If then we denote by the suffix 0 four functions Φ_0, Ψ_0, i_0, e_0 related to each other as in the analysis of the preceding section, the solution of (25a) and (25b) for the present case will be

$$\left.\begin{aligned} \Phi(x, y, z, t) = \frac{\Phi_0(x, y)}{e_0}\, e(z, t), \\ \Psi(x, y, z, t) = \frac{\Psi_0(x, y)}{i_0}\, i(z, t). \end{aligned}\right\} \quad . \quad . \quad . \quad (25c)$$

But, by equation (19), p. 203,

$$\frac{\Psi_0}{i_0} = \frac{K}{c}\,\frac{\Phi_0}{e_0},$$

so that

$$\Psi(x, y, z, t) = \frac{K}{c}\,\frac{\Phi_0(x, y)}{e_0}\, i(z, t). \quad . \quad . \quad . \quad (25d)$$

We have still to satisfy the four remaining equations (24a, b, d, e). If we introduce Φ and Ψ from (25a, c, d), we obtain from (24a) or (24b), and (24d) or (24e),

$$\frac{\partial e}{\partial t} + \frac{\partial i}{\partial z} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26a)$$

$$\frac{\mu K}{c^2}\,\frac{\partial i}{\partial t} + \frac{\partial e}{\partial z} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26b)$$

On eliminating e we find for i the wave equation

$$\frac{\partial^2 i}{\partial t^2} = \frac{c^2}{\mu K}\frac{\partial^2 i}{\partial z^2};$$

and similarly

$$\frac{\partial^2 e}{\partial t^2} = \frac{c^2}{\mu K}\frac{\partial^2 e}{\partial z^2}.$$

Writing a for the wave velocity $c/\sqrt{(\mu K)}$, we have therefore the general solution

$$e = f_1(t - z/a) + f_2(t + z/a), \quad . \quad . \quad . \quad (26c)$$

$$i = af_1(t - z/a) - af_2(t + z/a), \quad . \quad . \quad . \quad (26d)$$

containing two arbitrary functions f_1 and f_2.

By making use of the concepts of "self-inductance" and "capacity", we can obtain the fundamental equations (26a) and (26b) for waves along wires in another way, as follows.

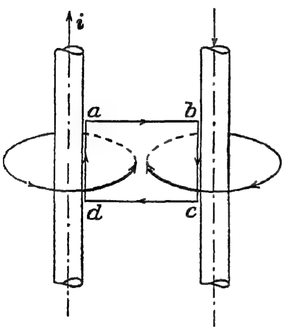

Fig. 4

In the first place, we note that the charge on the element dz of a wire can only change if the current entering differs in amount from the current leaving. This gives the "equation of continuity" $\partial e/\partial t = -\partial i/dz$, in agreement with (26$a$).

Next, we may apply Faraday's law of induction to a strip of breadth dz lying between the two conductors (the area $abcd$ of fig. 4). By the definition of (20), p. 204, the flux of induction through this strip is equal to $cL_e i\, dz$. Now the line integral $\int E_s\, ds$, along the path $abcd$, is

$$(\Phi_a - \Phi_b) - (\Phi_d - \Phi_c) = \frac{\partial(\Phi_1 - \Phi_2)}{\partial z}\, dz.$$

The line integral is equal to the rate of decrease of $L_e i$, so that

$$\frac{\partial(\Phi_1 - \Phi_2)}{\partial z} = -L_e\frac{\partial i}{\partial t}.$$

When we introduce the capacity $C = e/(\Phi_1 - \Phi_2)$, this becomes

$$\frac{1}{C}\frac{\partial e}{\partial z} + L_e\frac{\partial i}{\partial t} = 0.$$

Noting that $CL_e = \mu K/c^2$ (p. 204), we see that this equation is the same as (26b).

Consider in particular a wave of frequency $\omega/2\pi$, advancing in the positive z-direction. From (26c, d) we have the solution

$$\Phi = \Phi_0 \sin\omega(t - z/a),$$
$$i = Ca\, \Phi_0 \sin\omega(t - z/a),$$

where, for brevity, we have written Φ for $\Phi_1 - \Phi_2$.

The field $\mathbf{E}$, $\mathbf{H}$ *thus obtained differs from that of a plane wave in a homogeneous medium in this respect only, that in the wave plane* (xy *plane*) *the direction of polarization and the intensity are both functions of position.* It is only in the neighbourhood of the wires that the intensity is sensibly different from zero; and at any point of the surface of either wire the electric vector is perpendicular to the wire. The picture suggested is that of a wave sliding along the double circuit. The general course of the field for the plane wave has already been depicted in fig. 4, p. 131.

Among the multifarious applications of waves along wires, we shall only briefly consider the following example.

Let the twin circuit extend from the plane $z = -l$ to $z = 0$. At the latter plane let the circuit be closed by an ohmic resistance R. At the other plane $z = -l$, let there be a source of alternating current of frequency $\omega/2\pi$. In order to get a convenient view of the resulting distribution of pressure and current, we first, from (26c) and (26d), write down the general solution of frequency $\omega/2\pi$:

$$\Phi = pe^{j\omega(t-z/a)} + q'e^{j\omega(t+z/a)},$$
$$\frac{i}{Ca} = pe^{j\omega(t-z/a)} - q'e^{j\omega(t+z/a)},$$

where p and q' are any complex numbers. We may, however, without loss of generality, take p to be real, and put $q' = qe^{j\beta}$, where q and β are also real numbers. Further, we put for brevity

$$\frac{2\omega z}{a} = \frac{4\pi z}{\lambda} = \zeta,$$

i.e. we measure lengths along the twin circuit with $\lambda/4\pi$ as unit; ζ increases by 2π when z increases by half a wave-length. Our solution now runs

$$\left.\begin{aligned} \Phi &= e^{j\omega(t-z/a)}\left\{p + qe^{j(\beta+\zeta)}\right\}, \\ \frac{i}{Ca} &= e^{j\omega(t-z/a)}\left\{p - qe^{j(\beta+\zeta)}\right\}. \end{aligned}\right\} \quad . \quad . \quad . \quad (27)$$

We can now construct the vector diagram giving Φ and i for an arbitrary section z. For this purpose the common phase-factor $e^{j\omega(t-z/a)}$ is of no consequence. The meaning of the three constants p, q, β is

evident from fig. 5. Describe a circle of radius q, with centre at B, where $OB = p$. (The centre B is not marked in fig. 5.) Then turn the diameter which passes through O, viz. GH, through the angle $\beta + \zeta$, so that it takes up the position FE. (In the figure ζ, as in our example, is negative.) Then, as follows from (27), the quantities Φ and i/Ca are correctly represented by the vectors OF and OE both in amplitude and in difference of phase, for the section $z = \zeta\lambda/4\pi$. The state of affairs at any point along the circuit can accordingly be made out very easily by simply turning the diameter through the proper angle.

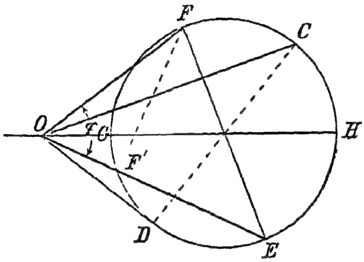

Fig. 5

This general description may now be particularized according to the condition prescribed at the far end of the circuit, which in our case is: at $\zeta = 0$, there is an ohmic resistance R, so that $\Phi = iR$ there. The equality of phase between Φ and i for $\zeta = 0$ requires $\beta = 0$. Also, by (27), for this boundary condition at $\zeta = 0$, we have

$$\frac{p-q}{p+q} = \frac{i}{\Phi Ca} = \frac{1}{RCa}.$$

Hence the ratio of OG to OH is also determined, and with it (except for a numerical factor) the circumstances at every point of the circuit.

We may note the following limiting cases.

(i) $R = \infty$. *Circuit open at the end.* Here $q = p$, so that the circle passes through O; and Φ, i have at every point the phase-difference 90°. For $z = 0, \frac{1}{2}\lambda, \lambda, \ldots$, we have $i = 0$, and for $z = \frac{1}{4}\lambda, \frac{3}{4}\lambda, \frac{5}{4}\lambda, \ldots$, $\Phi = 0$. (*Standing waves.*)

(ii) $R = 1/Ca = La$. This gives $q = 0$. The reflected wave disappears altogether, the resistance completely absorbing the incident wave. In this case i/Ca and Φ have at any given point equal values and the same phase.

(iii) $R = 0$. *Short-circuit at the end.* Here $q = -p$. This case only differs from that in which $R = \infty$ in respect that i and Φ have to be interchanged in the results as stated.

The *energy transmitted* along the circuit has of course the same value for every section, as may be seen at once from fig. 5. It is given in fact, except for the factor $\frac{1}{2}Ca$, if the angle EOF is ϕ, by

$$OF \,.\, OE \cos\phi = OF' \,.\, OE = OG \,.\, OH.$$

For example, it is always zero if the circuit is closed at the end by seld-inductances or capacities only, without ohmic resistance. For in that case the phase angle ϕ at the end of the circuit is $\pm 90°$, and this requires $|q| = p$. The circle in the diagram must therefore always pass through the origin O.

8. Waves along Wires of Finite Resistance.

It may now be asked: what modifications must be made in the above picture of the propagation of waves in perfect conductors, when the ohmic resistance which is always present is taken into account? The question is one of fundamental importance for practical telegraphy. It is obvious from the outset that the Joule heat which is necessarily developed will cause a *damping* of the waves. Moreover, as we shall see, both damping and wave velocity will depend on the frequency, and this will introduce *distortion* of the signals transmitted, those representing speech, for example.

From the standpoint of Maxwell's theory ohmic resistance in the wire implies a longitudinal component in the electric field; we cannot now put $E_z = 0$. Of the eight equations (24), p. 206, it follows that only the four (*a*), (*b*), (*f*), (*h*) remain as they were, while the other four require to be supplemented by a term containing the derivative of E_z with respect to one of the four variables x, y, z, t. Further, the field now penetrates into the conductor, so that we can no longer confine ourselves to the consideration of the insulator. On this account the problem becomes so complicated that a rigorous general solution cannot be given. Fortunately, however, in all cases of practical importance the circumstances are such that we can simplify the problem by making certain assumptions with respect to the order of magnitude of E_z. These can be expressed briefly—though not absolutely correctly—in the form:

$$\left.\begin{array}{l}\text{electric transverse field in the insulator}\\ \gg \text{longitudinal field}\\ \gg \text{transverse field in the metal.}\end{array}\right\} \quad . \quad . \quad . \qquad (28)$$

These relations are to be understood in the following sense.

(*a*) Since equations (24*f*, *h*) still hold, we can, as before, represent the transverse field by

$$E_x = -\frac{\partial\Phi}{\partial x},\quad H_x = \frac{\partial\Psi}{\partial y},$$

$$E_y = -\frac{\partial\Phi}{\partial y},\quad H_y = -\frac{\partial\Psi}{\partial x}.$$

(With respect to the assumption $H_z = 0$, see (*b*) below.)

Then the equations $\operatorname{div} \mathbf{E} = 0$, and $(\operatorname{curl} \mathbf{H})_z = \dfrac{K}{c}\dfrac{\partial E_z}{\partial t}$ give,

$$\left.\begin{aligned} \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} &= \frac{\partial E_z}{\partial z}, \\ -\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\right) &= \frac{K}{c}\frac{\partial E_z}{\partial t}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (29)$$

Thus Φ and Ψ do not now satisfy (25*b*) of the preceding section (p. 207). If, however, $\partial E_z/\partial z$ and $\partial E_z/\partial t$ are sufficiently small, we may for *practical* purposes take it that equations (25*b*) still hold good.

We shall begin by dealing with the problem to this approximation, and afterwards show by means of the solution we obtain that the assumption (28) is, in the sense mentioned, actually justified. (Cf. equation (33), p. 216.)

(*b*) The second assumption in (28), that $E_z \gg$ the transverse field in the metal, is necessary if, as we shall always do, we are to put $H_z = 0$. Otherwise, in fact, the current would have a transverse component, which in turn would give rise to a longitudinal component of $\mathbf{H}$.

Since equations (17), p. 203, for the charge e on unit length of the wire, and for the current i, continue to hold in all cases, the neglect of $\partial E_z/\partial z$ and $\partial E_z/\partial t$ in (29) implies that the transverse field in the *insulator* will agree exactly with that for the steady case, as calculated in § 6, p. 201. In particular, therefore, the quantities C and L_e, which were introduced there, can be taken over without change. Thus we have as before the equations

$$C(\Phi_1 - \Phi_2) = e,$$
$$cL_e i = \mu(\Psi_1 - \Psi_2)$$

for the charge, and for the flux of induction through a strip of breadth 1 extending from wire to wire. We can now, exactly as in the preceding section (fig. 4, p. 208), determine the law of propagation by applying the equation of continuity and the law of induction. But the integral round the rectangle *abcd* now includes contributions from the two sides *bc* and *da*, viz. $-E_{z_2}$ and E_{z_1}, so that we obtain

$$\left.\begin{aligned} \frac{\partial e}{\partial t} + \frac{\partial i}{\partial z} &= 0 \\ \frac{\partial(\Phi_1 - \Phi_2)}{\partial z} + E_{z_1} - E_{z_2} &= -L_e\frac{\partial i}{\partial t}. \end{aligned}\right\} \quad . \quad . \quad . \quad (30)$$

Here E_{z_1} and E_{z_2} denote the values of E_z at the surface of the conductor, and also in fact the surface values within the metal (the tangential component of $\mathbf{E}$ being continuous). Up to this point the field

within the wire has been disregarded. By taking it into account, the quantities E_{z_1} and E_{z_2} can be eliminated from the equation last written. This is done by remarking that within the wire the circumstances are the same as those already considered in connexion with the skin effect (p. 196). Hence equation (12), p. 198, i.e. in effect

$$E_{z_1} = R_1 i + L_{i_1} \frac{\partial i}{\partial t},$$

can be extended at once to the present case. Here, it is true, R and L_i are not constants; they depend essentially on the frequency ω of the alternating field (p. 199). The present solution, like the other, is therefore applicable to a sinusoidal field only.

If we now write

$$R_1 + R_2 = R, \text{ and } L_{i_1} + L_{i_2} = L_i,$$

so that R is the ohmic resistance and L_i the *internal* self-inductance of our double circuit, the second equation of (30) becomes

$$\frac{1}{C}\frac{\partial e}{\partial z} + Ri + L_i \frac{\partial i}{\partial t} = - L_e \frac{\partial i}{\partial t},$$

so that, by the first of (30),

$$\frac{\partial^2 i}{\partial z^2} = CR \frac{\partial i}{\partial t} + C(L_i + L_e) \frac{\partial^2 i}{\partial t^2}. \quad . \quad . \quad . \quad . \quad (31)$$

For a wave of frequency $\omega/2\pi$, we therefore have, if we write L for the total self-inductance $L_i + L_e$,

$$\frac{\partial^2 i}{\partial z^2} = (j\omega CR - \omega^2 CL)i.$$

We can satisfy this equation by putting

$$i = i_0 \, e^{j(\omega t - \gamma z)}.$$

This gives for γ the equation

$$\gamma^2 = \omega^2 CL - j\omega CR. \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

If we split up γ into a real and an imaginary part,

$$\gamma = \alpha - j\beta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32a)$$

we have

$$\left.\begin{aligned} \left(\frac{\alpha}{\omega}\right)^2 &= \frac{CL}{2} + \tfrac{1}{2}\sqrt{\{(CL)^2 + (CR/\omega)^2\}}, \\ \left(\frac{\beta}{\omega}\right)^2 &= -\frac{CL}{2} + \tfrac{1}{2}\sqrt{\{(CL)^2 + (CR/\omega)^2\}}. \end{aligned}\right\} \quad . \quad . \quad (32b)$$

With these values our solution becomes

$$i = i_0\, e^{-\beta z} e^{j(\omega t - \alpha z)}.$$

Hence ω/α is the phase velocity of the wave, and $1/\beta$ is that distance after traversing which the amplitude of the wave is damped to the e*th part of its original value.*

In order to throw into relief the effect of the field which penetrates into the metal, we split up L again in (32) into its components L_e and L_i, and use the relation (p. 204), which holds here also,

$$CL_e = \frac{1}{a^2},$$

where a is the velocity of light in the dielectric. Then (32) may be written in the form

$$\left(\frac{\gamma}{\omega}\right)^2 = \frac{1}{a^2}\left\{1 + \frac{L_i}{L_e} - j\frac{R}{\omega L_e}\right\}. \quad . \quad . \quad . \quad (32c)$$

The numbers
$$l = \frac{L_i}{L_e}, \quad \text{and} \quad r = \frac{R}{\omega L_e} = \frac{RCa^2}{\omega} \quad . \quad . \quad . \quad (32d)$$

may be taken as characterizing the "inductive" and the "ohmic" contributions of the metal to the action of the field as a whole. If these numbers are both small compared with unity, the expansion

$$\sqrt{(1 + l - jr)} = 1 + \tfrac{1}{2}(l - jr) - \tfrac{1}{8}(l - jr)^2 + \dots$$

gives, for the components α and β of $\alpha - j\beta = \gamma = (\omega/a)\sqrt{(1 + l - jr)}$,

$$\frac{\alpha}{\omega} = \frac{1}{a}\{1 + \tfrac{1}{2}l + \tfrac{1}{8}(r^2 - l^2) + \dots\},$$

$$\frac{\beta}{\omega} = \frac{1}{a}\{\tfrac{1}{2}r - \tfrac{1}{4}lr\}.$$

To a first approximation, the fractional fall $(a - V)/a$ in the velocity of the wave is $\frac{1}{2}l$ or $\frac{1}{8}r^2$, whichever of these numbers is the greater. The interpretation of r is that the amplitudes are diminished in the ratio $e^{\pi r} : 1$ when the wave traverses a wave-length $\lambda = 2\pi a/\omega$. The number r therefore determines the damping; its order of magnitude is found as follows.

We have, as in (32d), $r = \dfrac{R}{\omega L_e} = \dfrac{RCa^2}{\omega}$.

If R is measured in ohms per cm., and we put $a = 3 \times 10^{10}$ cm./sec., then

$$r = \frac{\lambda(\text{cm.})}{2\pi} C \frac{R(\text{ohms/cm.})}{9 \times 10^{11}} \times 3 \times 10^{10} = \frac{C}{2\pi}\lambda(\text{cm.}) R(\text{ohms/cm.}) \times \tfrac{1}{30}.$$

For copper wires of 1 sq. mm. section, neglecting the skin effect, we have

$$R = 2 \times 0{\cdot}00017 = 34 \times 10^{-5},$$

so that
$$r = \frac{C}{2\pi} \times 1{\cdot}1\lambda\,(\text{km.}),$$

where $\lambda\,(\text{km.}) = \lambda\,(\text{cm.}) \times 10^{-5}$ denotes the wave-length in kilometres. Since C is of the order of magnitude 1, it follows that for the frequencies usually occurring ($\omega/2\pi = 3000$; $\lambda \approx 3 \times 10^{10}/(3 \times 10^3) = 10^7 = 100$ km.) the number r is not small, but so great compared with 1 that in this case 1 and l are relatively negligible. We have therefore the approximation $r \gg 1$, or

$$\frac{\gamma}{\omega} = \frac{\sqrt{r}}{a}\sqrt{-j} = \sqrt{\left(\frac{RC}{2\omega}\right)}\,(1-j) = \frac{\alpha - j\beta}{\omega}.$$

Hence
$$\frac{\alpha}{\omega} = \sqrt{\left(\frac{RC}{2\omega}\right)}, \quad \beta = \sqrt{\left(\frac{\omega RC}{2}\right)},$$

as might be obtained at once from (32a) and (32b), for the case $L\omega \ll R$.

Thus if the wave-length is decidedly greater than 1 km., we have the results:

$$\text{phase velocity} = \frac{\omega}{\alpha} = \sqrt{\left(\frac{2\omega}{RC}\right)},$$

$$\text{depth of penetration} = \frac{1}{\beta} = \sqrt{\left(\frac{2}{\omega RC}\right)}.$$

We have now obtained the solution of the problem, viz. $i = i_0 e^{j(\omega t - \gamma z)}$; it remains to determine the limits of its validity. For this purpose we must obtain an estimate of the order of magnitude of the error which we may have committed in neglecting the right sides of equations (29). Along with i, we of course know e and $E_z - E_{z_2}$ also, from (30), viz.

$$e = \frac{\gamma}{\omega}\,i,$$

$$E_{z_1} - E_{z_2} = i\left(-j\omega L_e + j\frac{\gamma^2}{\omega C}\right).$$

Since $L_e C = 1/a^2$, and, by (32c) and (32d), $\gamma^2 a^2/\omega^2 = 1 + l - jr$, the latter equation becomes

$$E_{z_1} - E_{z_2} = i \,.\, j\omega L_e(l - jr).$$

We now return to equation (29), p. 212, which we shall deal with by considering the values of

$$\int \frac{\partial \Phi}{\partial n}\,ds \quad \text{and} \quad \int \frac{\partial \Psi}{\partial n}\,ds$$

taken round a curve encircling the first conductor at a distance d. For the first integral we have

$$\int \frac{\partial \Phi}{\partial n}\, ds = \frac{4\pi}{K}\, e + \int\!\!\int \frac{\partial E_z}{\partial z}\, dS,$$

where dS denotes an element of the area bounded by the curve. E_z has its greatest value E_{z_1} at the surface of the wire. The area $\iint dS$ is of the order of magnitude d^2. The contribution from $\partial E_z/\partial z$ to the flux of force $\iint (\partial\Phi/\partial n)\, dS$ can therefore be regarded as unimportant if

$$e \gg d^2 \frac{\partial E_{z_1}}{\partial z},$$

where d is a transverse dimension of the circuit, say the distance between the wires.

Again, the integral

$$\frac{c}{4\pi} \int \frac{\partial \Psi}{\partial n}\, ds = \frac{c}{4\pi} \int (\mathbf{H}\, ds)$$

is determined by the sum of the conduction current i and the total displacement current

$$\frac{K}{4\pi} \int\!\!\int \frac{\partial E_z}{\partial t}\, dS.$$

Accordingly, the neglect of the displacement current in (29) will be justified if

$$i \gg d^2 \frac{\partial E_{z_1}}{\partial t}.$$

If in these two results we insert the values for e and E_{z_1} as noted above, we obtain the same condition in both cases, viz.

$$1 \gg d^2\omega^2 L_e(jr - l), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

or, in view of the meaning of r and l,

$$1 \gg d^2\omega R, \quad \text{and} \quad 1 \gg d^2\omega^2 L_i.$$

We consider the first of these two conditions under the most unfavourable assumption for R, i.e. for the case of a strong skin effect. If b is the radius of the wires, then, by (14a) and (14b), p. 200,

$$R = R_0 \frac{b\sqrt{(\pi\omega\sigma)}}{c\sqrt{2}}; \quad R_0 = \frac{1}{\pi b^2 \sigma}.$$

Our condition therefore runs

$$1 \gg \sqrt{\left(\frac{\omega}{\sigma}\right)}\, \frac{\omega}{c}\, \frac{d^2}{b}.$$

It may be infringed by extremely large values of ω (waves too short), or by too small a wire-radius b. For short electric waves $\omega \approx 10^7$ sec.$^{-1}$, and for metals $\sigma \approx 10^{17}$ sec.$^{-1}$. With these values

$$\sqrt{\left(\frac{\omega}{\sigma}\right)}\frac{\omega}{c} \approx 10^{-5} \times \frac{10^{-3}}{3} = 3 \times 10^{-9}\ (1/\text{cm.}).$$

For an interval between the wires of 10 cm., we have therefore the condition for b

$$b/d^2 \gg 3 \times 10^{-9}\ (1/\text{cm.}), \quad \text{or} \quad b \gg 3 \times 10^{-7}\ \text{cm.},$$

a condition which in practice is satisfied as a matter of course.

9. The Complex Poynting Vector and the Equation of Telegraphy.

We shall now give another discussion of the problem of last section. This time we shall from the outset use the fact explicitly that the field variables only involve t and z by their containing the factor

$$e^{j(\omega t - \gamma z)}.$$

In Maxwell's equations we again put $H_z = 0$; and, if ϕ is any field function which occurs, we write

$$\frac{\partial \phi}{\partial t} = j\omega\phi, \quad \frac{\partial \phi}{\partial z} = -j\gamma\phi.$$

We shall not, however, confine our equations to the insulator, and shall therefore include the term $4\pi\sigma\mathbf{E}/c$ for the conduction current. Thus we have the equations

$$\left.\begin{aligned}
j\gamma H_y &= \frac{4\pi\sigma + jK\omega}{c} E_x, && (a)\\
-j\gamma H_x &= \frac{4\pi\sigma + jK\omega}{c} E_y, && (b)\\
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= \frac{4\pi\sigma + jK\omega}{c} E_z, && (c)\\
\frac{\partial E_z}{\partial y} + j\gamma E_y &= -\frac{j\mu\omega}{c} H_x, && (d)\\
-\frac{\partial E_z}{\partial x} - j\gamma E_x &= -\frac{j\mu\omega}{c} H_y, && (e)\\
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= 0. && (f)
\end{aligned}\right\} \quad . \quad . \quad . \quad (34)$$

We can satisfy (f), (a), and (b) by means of a *single* function Φ, putting

$$\left.\begin{aligned} E_x &= -\frac{\partial \Phi}{\partial x}, \quad H_x = \frac{K\omega - 4\pi j\sigma}{\gamma c}\frac{\partial \Phi}{\partial y}, \\ E_y &= -\frac{\partial \Phi}{\partial y}, \quad H_y = -\frac{K\omega - 4\pi j\sigma}{\gamma c}\frac{\partial \Phi}{\partial x}. \end{aligned}\right\} \quad . \quad . \quad (35)$$

For E_z we then find two equations, one from (c), viz.

$$E_z = -\frac{1}{j\gamma}\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right), \quad . \quad . \quad . \quad . \quad (36a)$$

and a second from (d) and (e). If we write for shortness

$$a'^2 = \frac{c^2}{\mu K - 4\pi j\mu\sigma/\omega}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (36b)$$

the second equation for E_z becomes

$$E_z = -\frac{1}{j\gamma}\left\{\gamma^2 - \left(\frac{\omega}{a'}\right)^2\right\}\Phi. \quad . \quad . \quad . \quad . \quad . \quad (36c)$$

From (36a) and (36c) we obtain for the complex function Φ the equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = \left\{\gamma^2 - \left(\frac{\omega}{a'}\right)^2\right\}\Phi, \quad . \quad . \quad . \quad (36d)$$

which, when integrated subject to the appropriate boundary conditions, must determine the complex wave-number γ. This integration, however, with arbitrary given values of a' for insulator and metals, is not an easy matter. The approximate treatment given in the preceding section amounts to putting the right side of (36d) $= 0$ in the *insulator*; and in the *metal* taking γ^2 as small compared with $(\omega/a')^2$, in accordance with the theory of the skin effect (p. 196). Consequently γ disappears from the differential equation altogether. It is determined at a later stage by means of the boundary conditions (continuity of the tangential components of **E** and **H**).

We shall now show how the results of the preceding section can be arrived at directly by means of the complex Poynting vector ((9a) and (9b), p. 196)

$$\mathbf{N}' = \frac{c}{8\pi}[\mathbf{E}\mathbf{H}^*].$$

For the z-component of this, we obtain at once from (34a, b)

$$\begin{aligned} N_z' &= \frac{c}{8\pi}(E_x H_y^* - E_y H_x^*) \\ &= \frac{c}{8\pi}\,\frac{j\gamma c}{4\pi\sigma + jK\omega}(H_x H_x^* + H_y H_y^*). \end{aligned}$$

Now $(H_x H_x^* + H_y H_y^*)/8\pi$ is equal to twice the magnetic energy density U_{mag}. By integrating N_z' over a plane area perpendicular to the wires, we therefore obtain a relation between the total flux F' of the Poynting vector in the direction of the double circuit, and the magnetic field energy per unit length in that direction. Since we are putting $K = 0$ in the metal, and $\sigma = 0$ in the insulator, this relation is

$$\left.\begin{aligned} F' = \iint N_z'\, dS = \frac{\gamma c^2}{K\omega} \,.\, 2\,(U_{\text{mag}} \text{ in insulator}) \\ + \frac{j\gamma c^2}{4\pi\sigma} \,.\, 2\,(U_{\text{mag}} \text{ in metal}). \end{aligned}\right\} \quad . \; . \; . \quad (37)$$

We now take two cross-sections of the double circuit 1 cm. apart. Then $-\partial F'/\partial z$ is the part of the "complex stream of energy" N' which is left behind between these two sections. But, by (9b), p. 196,

$$-\frac{\partial F'}{\partial z} = \text{Joule heat} + 2j\omega\,(\text{magnetic energy} - \text{electric energy}). \quad (37a)$$

We can work this out approximately, as follows.

In calculating the electric field energy we neglect the longitudinal component E_z of $\mathbf{E}$. Then, by (34a, b),

$$K\mathbf{EE}^* = \frac{K\gamma\gamma^* c^2}{(K\omega)^2 + (4\pi\sigma)^2}\, \mathbf{HH}^*.$$

Further, it is only in the insulator that we need consider the electric energy, so that

$$U_{\text{el}} = \frac{\gamma\gamma^* c^2}{K\omega^2}\,(U_{\text{mag}} \text{ in insulator}).$$

Again, we bring in the resistance, as also the external and internal self-inductance, by the equations:

$$\begin{aligned} \text{Joule heat} \qquad &= \tfrac{1}{2} R i i^*, \\ U_{\text{mag}} \text{ (in insulator)} &= \tfrac{1}{4} L_e i i^*, \\ U_{\text{mag}} \text{ (in metal)} \quad &= \tfrac{1}{4} L_i i i^*. \end{aligned}$$

Then in (37a) we carry out the z-differentiation. Since F' depends on z in virtue of the factor

$$e^{-j\gamma z}\, e^{j\gamma^* z} = e^{-j(\gamma - \gamma^*)z},$$

we have
$$-\frac{\partial F'}{\partial z} = j(\gamma - \gamma^*)\, F'.$$

We have now to insert the value of F' from (37). In doing so, we can

neglect the second part (that relative to the metal). Thus, finally, from (37*a*), on rejecting the factor $\frac{1}{2}ii^*$, we find

$$j\gamma(\gamma - \gamma^*)\frac{c^2}{K\omega}L_e = R + j\omega\left(L_i + L_e - \frac{\gamma\gamma^* c^2}{K\omega^2}L_e\right),$$

or
$$\frac{c^2}{K\omega^2}L_e\{\gamma(\gamma-\gamma^*) + \gamma\gamma^*\} = -\frac{jR}{\omega} + L_i + L_e,$$

so that
$$\gamma^2 = \frac{K\omega^2}{c^2}\left(1 + \frac{L_i}{L_e} - j\frac{R}{\omega L_e}\right),$$

which agrees exactly with the result (32*c*), p. 214, obtained and discussed in last section.

10. The General Electrodynamic Potentials.

In this section and the next, we set ourselves the problem of calculating the field which is produced by an assigned distribution of charge and current density, which varies with the time in any given manner.

Let the current density be i(x, y, z, t), *and the density of charge* ρ(x, y, z, t), *these being given functions of position and of the time.*

We shall confine the discussion to the case of propagation of the field in empty space, so that we put $K = \mu = 1$. The field is therefore defined by the equations

$$\left.\begin{aligned}\operatorname{curl}\mathbf{H} - \frac{1}{c}\dot{\mathbf{E}} = \frac{4\pi}{c}\mathbf{i},\quad (a)\\ \operatorname{div}\mathbf{E} = 4\pi\rho,\quad (b)\end{aligned}\right\}\qquad \left.\begin{aligned}\operatorname{curl}\mathbf{E} + \frac{1}{c}\dot{\mathbf{H}} = 0,\quad (c)\\ \operatorname{div}\mathbf{H} = 0.\quad (d)\end{aligned}\right\}. \qquad (38)$$

From (*a*) and (*b*) it follows in the first place that **i** and ρ cannot be quite arbitrarily prescribed, but that the equation of continuity

$$\operatorname{div}\mathbf{i} + \dot{\rho} = 0$$

must be satisfied everywhere and always. We can satisfy (38*d*) identically by introducing the vector potential **A**, so that

$$\mathbf{H} = \operatorname{curl}\mathbf{A}. \qquad (39a)$$

With this value of **H**, (38*c*) states that $\mathbf{E} + \dot{\mathbf{A}}/c$ is irrotational. Hence this equation is satisfied if we put

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}} - \operatorname{grad}\phi. \qquad (39b)$$

When we insert these values of $\mathbf{E}$ and $\mathbf{H}$ in (38*a*, *b*) they become

$$\text{curl curl } \mathbf{A} + \frac{1}{c^2}\ddot{\mathbf{A}} + \frac{1}{c}\text{grad } \dot{\phi} = \frac{4\pi}{c}\mathbf{i},$$

$$-\frac{1}{c}\text{div } \dot{\mathbf{A}} - \Delta\phi = 4\pi\rho.$$

Now (39*a*) only specifies the curl of the vector $\mathbf{A}$. Its sources are still at our disposal. We define them by laying down the condition

$$\text{div } \mathbf{A} = -\frac{1}{c}\dot{\phi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39c)$$

Hence, when we take account of (33), p. 36, the two preceding equations become

$$\Delta\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{i}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (40a)$$

$$\Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\rho. \quad . \quad . \quad . \quad . \quad . \quad . \quad (40b)$$

For the case of the steady field these equations pass immediately into the equations of electrostatics and of steady currents, which have already been treated in detail (cf. § 7, p. 24; § 11, p. 37). The familiar statical equations involving the operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ are supplemented in (40*a*, *b*) by terms in $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$, which take explicit account of the propagation of the field in time.

The general integral of (40*a*, *b*) can be put into a form very similar to the one which holds for steady fields. The general integral, as we shall immediately verify, is as follows:

$$\mathbf{A}(x, y, z, t) = \frac{1}{c}\iiint \frac{\mathbf{i}(\xi, \eta, \zeta, t - r/c)}{r}\, d\xi\, d\eta\, d\zeta, \quad . \quad (41a)$$

$$\phi(x, y, z, t) = \frac{1}{c}\iiint \frac{\rho(\xi, \eta, \zeta, t - r/c)}{r}\, d\xi\, d\eta\, d\zeta, \quad . \quad (41b)$$

where $$r = \sqrt{\{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}}.$$

These equations state that the contribution of the element of volume $d\xi\, d\eta\, d\zeta$ to the potentials at the field point (x, y, z) at time t differs from that in the statical case in this *only*, that the values of the densities of charge and current which are to be inserted in the integrand are the values at the point (ξ, η, ζ), not at the time t, but at the earlier time $t - r/c$. *The contributions* $\mathbf{i}/c\mathrm{r}$ *and* ρ/r *which a source makes to the potentials at a point in the field do not arrive at that point till after a time* r/c. For this reason $\mathbf{A}$ and ϕ are often called *retarded potentials.*

We have still to satisfy ourselves that (41a, b) do actually give a solution of (40a, b). It will be sufficient to prove this for (41b). To do so, we divide the integration space into two parts V_1 and V_2, where V_1 is a very small volume containing the field point (x, y, z), and V_2 is the whole of the rest of the space. We then divide ϕ into the parts contributed by these volumes, viz.

$$\phi = \phi_1 + \phi_2 = \int_{V_1} \frac{\rho(t - r/c)}{r}\, dV + \int_{V_2} \frac{\rho(t - r/c)}{r}\, dV.$$

In the integral over the small volume V_1 the retardation is obviously of no consequence, so that in this integral we can replace $\rho(t - r/c)$ by $\rho(t)$ simply. But the integral is then precisely the same as in the statical case. We therefore have (p. 24)

$$\Delta\phi_1 = -\, 4\pi\rho(x, y, z, t).$$

Further, for a function $f(r)$ depending on r only (so far as the space variables are concerned), we have always (p. 43)

$$\Delta f(r) = \frac{1}{r}\, \frac{\partial^2}{\partial r^2}(rf),$$

so that
$$\Delta\phi_2 = \int_{V_2} \frac{1}{r}\, \frac{\partial^2}{\partial r^2}\, \rho(\xi, \eta, \zeta, t - r/c)\, d\xi\, d\eta\, d\zeta.$$

But, for any function F of $(t - r/c)$,

$$\frac{\partial^2 F}{\partial r^2} = \frac{1}{c^2}\, \frac{\partial^2 F}{\partial t^2}.$$

Hence

$$\Delta\phi_2 = \frac{1}{c^2} \iiint \frac{1}{r}\, \frac{\partial^2}{\partial t^2}\, \rho(\xi, \eta, \zeta, t - r/c)\, d\xi\, d\eta\, d\zeta = \frac{1}{c^2}\, \frac{\partial^2\phi}{\partial t^2}.$$

We have therefore verified that

$$\begin{aligned}\Delta\phi &= \Delta\phi_1 + \Delta\phi_2,\\ &= -\, 4\pi\rho + \frac{1}{c^2}\, \frac{\partial^2\phi}{\partial t^2}.\end{aligned}$$

Accordingly, the function ϕ defined in (41b) does actually satisfy the differential equation (40b).

The above solution of Maxwell's equations is perfectly general. In next section we shall discuss it in detail for the special case of the spherical waves which are sent out by an oscillating dipole.

11. Hertz's Solution.

In this section we shall specially consider the field due to that class of varying currents and charges, in which the whole generating system (the transmitter, or "sender") is concentrated within a small region in the neighbourhood of the origin of co-ordinates. Physically, this means that the dimensions of the sender are to be small compared to the wave-length λ of the emitted radiation, and small also compared to the distance r at which we investigate the field of the sender. Hence the potentials $\mathbf{A}$ and ϕ, from which the field is deduced by means of (39a, b), satisfy the following equations everywhere except in the immediate neighbourhood of the origin:

$$\Delta \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0, \quad . \quad . \quad . \quad . \quad (42a)$$

$$\Delta \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (42b)$$

$$\operatorname{div} \mathbf{A} + \frac{1}{c}\dot{\phi} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (42c)$$

The simplest solution for our purpose is obtained by imposing the condition that $\mathbf{A}$ is to depend only on the distance r from the origin, and that it is to have the same direction everywhere. We take the axis of z in this direction, and therefore put tentatively

$$A_x = 0,\ A_y = 0,\ \Delta A_z - \frac{1}{c^2}\ddot{A}_z = 0.$$

Since A_z is to depend on r only, we have

$$\Delta A_z = \frac{1}{r}\frac{\partial^2 (rA_z)}{\partial r^2},$$

so that a solution is

$$A_x = 0,\ A_y = 0,\ A_z = \frac{1}{cr}\dot{f}(t - r/c), \quad . \quad . \quad . \quad (43)$$

where f is a function, in the first instance arbitrary, of the single argument $t - r/c$. Equation (42c) now practically defines ϕ also:

$$\frac{1}{c}\dot{\phi} = -\operatorname{div} \mathbf{A} = -\frac{\partial A_z}{\partial z}.$$

But $$\frac{\partial A_z}{\partial z} = \frac{\partial A_z}{\partial r}\frac{\partial r}{\partial z} = -\left\{\frac{1}{c^2 r}\ddot{f}(t - r/c) + \frac{1}{cr^2}\dot{f}(t - r/c)\right\}\frac{z}{r}.$$

We may therefore take

$$\phi = \left\{\frac{1}{cr}\dot{f}(t - r/c) + \frac{1}{r^2}f\left(t - \frac{r}{c}\right)\right\}\frac{z}{r}. \quad . \quad . \quad (43a)$$

This value of ϕ satisfies (42b) also, as we may see at once by comparing its functional form with that of $\partial A/\partial z$, which obviously satisfies the wave equation.

To get a preliminary idea of the physical meaning of this solution, let us consider its behaviour at points so near the sender that we can neglect the retardation r/c in comparison with t, and the term which has r in the denominator in comparison with the one which has r^2. If, for example, as is usually the case in applications, f is a periodic function of the time, or

$$f = f_0 \sin 2\pi\nu(t - r/c) = f_0 \sin 2\pi(\nu t - r/\lambda),$$

we shall have, by (43a),

$$\phi = f_0 \left\{ \frac{1}{cr} 2\pi\nu \cos 2\pi(\nu t - r/\lambda) + \frac{1}{r^2} \sin 2\pi(\nu t - r/\lambda) \right\} \frac{z}{r},$$

or $$\phi = \frac{f_0}{r\lambda} \left\{ 2\pi \cos 2\pi(\nu t - r/\lambda) + \frac{\lambda}{r} \sin 2\pi(\nu t - r/\lambda) \right\} \frac{z}{r}.$$

If we now take $r \ll \lambda$, we see that we can in the first place cut out the term r/λ in $(\nu t - r/\lambda)$, and secondly reject the cosine term.

Hence, *in the neighbourhood of the sender* ($r \ll \lambda$), our solution has the form

$$A_z = \frac{\dot{f}}{cr}, \; \phi = \frac{f}{r^2} \frac{z}{r}.$$

But $\frac{f}{r^2}\frac{z}{r}$, i.e. $\left(-f\frac{\partial}{\partial z}\frac{1}{r}\right)$, is the electrostatic potential of an electric dipole, the moment f of which is in the direction of the positive z-axis. Also, by the Biot-Savart law, $\mathbf{A}$ is the vector potential of a current filament $i\, ds = \dot{f}$.

Now two metal balls, which carry charges $+e$ and $-e$, placed at a distance ds from each other, are equivalent to a dipole of moment $e\, ds$. If the charges change owing to the balls being connected by a spark gap or by a wire, then a current i (or de/dt) flows along ds. The condition $i\, ds = \dot{f}$ is therefore satisfied in this system.

Hence the solution (43) and (43a), for arbitrary values of r, which we now proceed to discuss, gives the *radiation from a dipole at* O, *which has the direction* Oz *and the moment* f = f (t).

As an aid to the discussion of the values of $\mathbf{E}$ and $\mathbf{H}$, which are now given by (39a, b), we introduce polar co-ordinates r, θ, α, taking Oz in the direction $\theta = 0$ (fig. 6). We then have, first,

$$\left.\begin{aligned} A_r = \frac{1}{cr}\dot{f}(t - r/c)\cos\theta, \; A_\theta = -\frac{1}{cr}\dot{f}(t - r/c)\sin\theta, \; A_\alpha = 0, \\ \phi = \left\{\frac{1}{cr}\dot{f}(t - r/c) + \frac{1}{r^2}f(t - r/c)\right\}\cos\theta. \end{aligned}\right\} \quad (43b)$$

Now $\mathbf{H} = \operatorname{curl} \mathbf{A}$, and therefore, by (38$f$), p. 43,

$$\left.\begin{aligned} H_r &= 0, \quad H_\theta = 0, \\ H_\alpha &= \frac{\sin\theta}{r}\left(\frac{\ddot{f}}{c^2} + \frac{\dot{f}}{cr}\right). \end{aligned}\right\} \quad \ldots \quad (44a)$$

Also, from (39b), p. 220,

$$E_r = -\frac{\ddot{f}}{c^2 r}\cos\theta + \left(\frac{\ddot{f}}{c^2 r} + \frac{2\dot{f}}{cr^2} + \frac{2f}{r^3}\right)\cos\theta,$$

$$\left.\begin{aligned} E_r &= 2\cos\theta\left(\frac{\dot{f}}{cr^2} + \frac{f}{r^3}\right), \\ E_\theta &= \frac{\ddot{f}}{c^2 r}\sin\theta + \left(\frac{\dot{f}}{cr^2} + \frac{f}{r^3}\right)\sin\theta. \end{aligned}\right\} \quad \ldots \quad (44b)$$

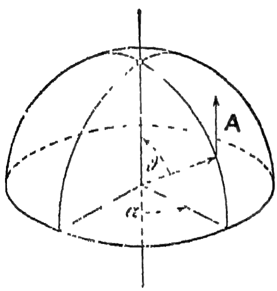

Fig. 6

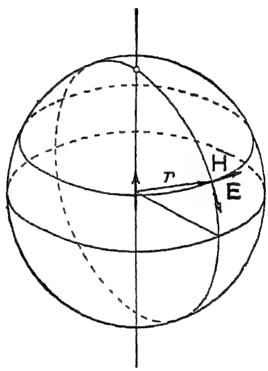

Fig. 7

These equations give the field of our sender at any distance. We shall discuss the two limiting cases $r \ll \lambda$ and $r \gg \lambda$.

In the neighbourhood of the sender ($r \ll \lambda$) the predominating terms are those with the highest powers of r in the denominator, i.e.

$$H_\alpha = \frac{\dot{f}}{cr^2}\sin\theta; \; E_r = \frac{2f}{r^3}\cos\theta; \; E_\theta = \frac{f}{r^3}\sin\theta. \quad . \quad (44c)$$

As might have been expected from results already found (p. 223), these represent the statical magnetic field of a current element $i\,ds = \dot{f}$, and the statical field of a dipole of moment f.

At a great distance from the sender ($r \gg \lambda$), on the other hand, the terms in $1/r$ are the only important ones. This is the region of the

$$\left.\begin{aligned} H_r = 0,\ H_\theta = 0,\ H_\alpha = \frac{\ddot{f}}{rc^2}\sin\theta, \\ E_r = 0,\ E_\theta = \frac{\ddot{f}}{rc^2}\sin\theta,\ E_\alpha = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad (45)$$

Accordingly, in the wave-zone the vectors **H** *and* **E** *are of equal magnitude, and perpendicular to each other and to the radius vector* **r**. Their magnitude diminishes from the equator to the pole in proportion to $\sin\theta$. We can now visualize the main features of our plane polarized wave, radiating from the sender in the direction **r**. *The electric vector vibrates in the meridian, the magnetic in the parallel of latitude.*

We shall also calculate the amount of the radiation U_r, i.e. the energy which the spherical wave described by (45) transmits per second across a spherical surface of radius r. The two parallels θ and $\theta + d\theta$ include between them the area $2\pi r^2 \sin\theta\, d\theta$. The Poynting vector being

$$\mathbf{N} = \frac{c}{4\pi}[\mathbf{EH}] = \frac{c}{4\pi}\left(\frac{\ddot{f}}{rc^2}\right)^2 \sin^2\theta,$$

we have
$$\begin{aligned} U_r &= \int\!\!\int N_n\, dS \\ &= \frac{c}{4\pi}\left(\frac{\ddot{f}}{rc^2}\right)^2 2\pi r^2 \int_0^\pi \sin^2\theta \,.\, \sin\theta\, d\theta. \end{aligned}$$

On putting $\cos\theta = x$, the integral becomes

$$\int_{-1}^{1} (1 - x^2)\, dx = \tfrac{4}{3},$$

so that
$$U_r = \frac{2}{3c^3}(\ddot{f})^2_{t-r/c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

The radiation U_r is, as it must be, only in so far independent of the radius r, that its value at time t is determined by the state of the sender at the previous time $t - r/c$.

It is also worthy of remark that it is the retardation in (43) and (43a), p. 223, which is responsible for the spherical character of the wave in (45). In fact, (45) is obtained at once if, when we are differentiating with respect to r, we only consider the r in the combination $t - r/c$, and therefore treat the "Coulomb" r in the denominator as constant. Conversely we obtain the result in the neighbourhood of the sender if we differentiate with respect to the Coulomb r only, and treat the "Maxwell" r in the numerator as constant.

12. The Radiation from a Linear Oscillator.

In periodic processes we are specially interested in the mean value of the radiation taken over a whole period. For this mean value, the retardation is clearly of no importance. To obtain the time average we can regard the sender either as an oscillating dipole, or as a current filament. The representation as a dipole is particularly suitable for the purposes of *atomic physics*, where the dipole moment of the individual atoms is the natural starting-point for the calculation of the radiation they emit. The view of the oscillator as a current filament, on the other hand, is the natural one to take when discussing the antennæ used in *wireless telegraphy*.

If the frequency of the dipole vibrations is ν, and their amplitude f_0, we have

$$f = f_0 \sin 2\pi\nu t$$

and

$$\overline{(\ddot{f})^2} = \tfrac{1}{2} f_0^2 (2\pi\nu)^4. \quad \ldots \ldots \quad (47a)$$

On the other hand, if l is the length of the dipole, and therefore also of the straight current filament joining the poles,

$$\dot{f} = il.$$

Hence, if the current oscillates with amplitude i_0, we have

$$i = i_0 \cos 2\pi\nu t; \quad \overline{i^2} = \tfrac{1}{2} i_0^2,$$

and

$$\ddot{f} = -2\pi\nu i_0 l \sin 2\pi\nu t,$$

so that

$$\overline{(\ddot{f})^2} = (2\pi\nu)^2 l^2 (\overline{i^2}). \quad \ldots \ldots \ldots \quad (47b)$$

We therefore obtain for the radiation U_r the two equivalent expressions

$$\overline{U}_r = \frac{f_0^2 (2\pi\nu)^4}{3c^3}, \quad \ldots \ldots \ldots \quad (48a)$$

and

$$\overline{U}_r = \frac{2}{3c^3} (2\pi\nu)^2 l^2 \overline{i^2} = \frac{8\pi^2}{3c} \frac{l^2}{\lambda^2} \overline{i^2}. \quad \ldots \quad (48b)$$

For *wireless telegraphy*, (48*b*) gives the radiation of an aerial of length l, for the wave-length λ and the effective aerial current $\sqrt{(\overline{i^2})}$. The oscillation generator has to supply, besides this radiation, the Joule heat

$$Q = R\overline{i^2}.$$

The factor multiplying $\overline{i^2}$ in (48*b*) is by analogy called the radiation resistance R_r of the antenna. The total rate of production of work by

the generator is then $(R + R_u)\,\overline{(i)}^2$, where R denotes the ohmic resistance (taking into account the skin effect) and R_u is given by

$$R_u = \frac{8\pi^2}{3c}\,\frac{l^2}{\lambda^2}\ \text{c.g.s. units,}$$

or
$$R_u = 790\,\frac{l^2}{\lambda^2}\ \text{ohms (l and λ in cm.).}$$

In the *emission of light by atoms,* where a continuous supply of energy is out of the question, (48*a*) indicates a rate of loss of energy carried off by the radiation, and consequently a *damping* of the emission; and this is capable of being observed spectroscopically, as a broadening of the spectral line. Also, (48*a*) gives the time of fading of the emission of light from a single atom, a quantity which, as the average duration of the excited state of an atom, has acquired fundamental importance in the quantum theory.

Up to this point we have considered only the special solution (43) of the general wave-equations (42). For an elementary current filament $i(t)\,ds$ it runs

$$\mathbf{A} = \frac{i(t - r/c)\,ds}{cr}. \qquad (49)$$

On going back, however, to the general expressions (41) for $\mathbf{A}$ and ϕ we see that, as regards the processes which take place in the wave-zone, this solution has a much more general character, provided only that the orders of magnitude of the quantities involved satisfy the relation:

$$\text{dimensions of sender} \ll \lambda \ll r. \qquad (50)$$

If the part of the sender which carries the current consists of a curved wire of cross-section S, and having ds as an element of its axis, then, in (41*a*), $\mathbf{i}\,d\xi\,d\eta\,d\zeta = i\,ds$, and we obtain

$$\mathbf{A} = \frac{1}{c}\int i(\xi, \eta, \zeta, t - r/c)\,\frac{1}{r}\,ds.$$

If (50) is satisfied, we may replace r here by the distance r_0 of the field point from the origin, so that

$$\mathbf{A} = \frac{1}{cr_0}\left\{\int i\,ds\right\}_{t-r/c}.$$

In particular, let the sender consist of two metallic pieces joined by a curved wire of any form, the capacity of the metal pieces being large

compared with that of the wire. In this case the current at any moment has the same value at every section of the wire. Hence (fig. 8)

$$\mathbf{A}(x, y, z, t) = \frac{1}{cr_0} i(t - r_0/c)\int_1^2 d\mathbf{s}. \quad . \quad . \quad . \quad (50a)$$

The only difference between this expression and (49) is that the line element $d\mathbf{s}$ appearing there is now replaced by the vector $\int_1^2 d\mathbf{s}$, which is represented by the *straight line* joining the two ends of the wire. It is therefore this straight line alone which determines the radiation emitted. It has its maximum value in the straight antennæ of wireless telegraphy (open oscillation circuits). It is vanishingly small when the wire conveying the current connects the coatings of a condenser of the usual type (closed oscillation circuits).

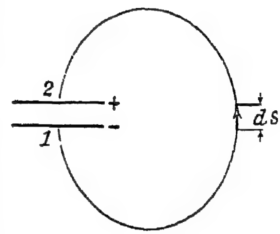

Fig. 8

This property constitutes the justification of our neglect of the radiation in our previous treatment (p. 176) of a circuit consisting of a capacity and a self-inductance. By means of the last formula we can in any concrete case estimate the error committed in neglecting the radiation in comparison with the Joule heat.

PART IV

ENERGY AND FORCES IN MAXWELL'S THEORY

CHAPTER XI

Thermodynamics of Field Energy

1. The Field Energy as Free Energy.

In our earlier discussions (pp. 151, 165) of the connexion between field energy and ponderomotive forces, we have tacitly assumed that when the changes in the field energy are sufficiently slow the mechanical and electrical work obtained is equal to the diminution of the field energy. From this assumption, and the assumed expression

$$\frac{1}{4\pi}\,\mathbf{E}\,d\mathbf{D}+\frac{1}{4\pi}\,\mathbf{H}\,d\mathbf{B}$$

for the change of the energy density in a given element of volume, we were able to deduce both the ponderomotive forces and the law of induction. In this energy balance, development of heat occurred only in the form of the "thermochemical" activity associated with conduction currents; that is to say, the heat was either Joule heat or Peltier heat. We shall now consider specially the interior of a *non-conducting dielectric*. With such a material, in our previous discussions, no development of heat occurred. The results found were therefore incorrect, unless in the course of the change of the dielectric polarization of individual elements no energy in the form of heat was either taken in or given out. But we must reckon with the possibility in the general case that in consequence of the polarization of a thermally insulated portion of matter, its temperature may change. It might appear indeed as if no difficulty in point of principle would arise on this account with respect to our former energy balance. For we could introduce the supplementary condition that all electrical (and also magnetic) changes of state are to take place *adiabatically*. This would mean, it is true, that in a non-homogeneous field the thermal con-

ductivity of the substance would require to be put equal to zero, since elements not equally strongly polarized will assume different temperatures. In point of fact, however, this artificial expedient for saving the former investigation would be most inconvenient and hardly practicable. In the first place, all substances do in fact possess a finite conductivity for heat. Experiments to test the theory will therefore never be carried out adiabatically, but as far as possible isothermally, i.e. with complete equalization of any differences of temperature that may occur. Further, the following point must also be noted: *the dielectric constant K is in general a function of the temperature.* As a rule it diminishes as the temperature increases. If, therefore, the temperature changes in the course of a process, K cannot be regarded as constant in the calculation of the field energy

$$\frac{1}{4\pi}\int_0^{\mathbf{E}} K\mathbf{E}\,d\mathbf{E},$$

even if the vectors $\mathbf{D}$ and $\mathbf{E}$ are strictly proportional to one another when the temperature is kept constant. The fundamental expression $K\mathbf{E}^2/8\pi$ for the energy density, in Maxwell's theory, can therefore by no means lay claim to general validity; and yet the whole of our discussion of ponderomotive forces is directly based upon the assumed correctness of this expression.

The only reasonable way of escape from the difficulty is to adopt the view that $K\mathbf{E}^2/8\pi$ does not represent the energy density at all, but the density of the *free energy*, in the thermodynamical sense. It is only by introducing this quantity, and making use of the second law of thermodynamics, that our earlier discussion can be completely justified. Consequently, we shall not accept the suggested proviso that our previous investigation is to be considered as referring to adiabatic processes; what we shall rather do is to lay it down as a definite understanding that the electric (and magnetic) polarizations dealt with in our earlier work are to take place *isothermally*. Thus, whenever the field is changing, such quantities of heat will be continuously absorbed or emitted, both in individual elements and in the system as a whole, as may be required to maintain the temperature constant.

We shall specially consider, with reference to § 2, p. 84, a cubic centimetre of a dielectric between the plates of a condenser. We can add energy to this portion of the dielectric in two ways: either by doing electrical work $\mathbf{E}\,d\mathbf{D}/4\pi$ (alteration of the charge on the condenser coatings), or by communicating a quantity of heat $d'Q$. Hence, if no other changes take place, the energy U of the system, condenser + dielectric, is increased by the amount

$$dU = d'Q + \frac{1}{4\pi}\mathbf{E}\,d\mathbf{D}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For an adiabatic change, we of course fall back upon the relation already discussed, $dU = (\mathbf{E}\, d\mathbf{D})/4\pi$. According to the second law there exists a function S of the state of the system, called the entropy, which is such that

$$d'Q = T\,dS, \quad \ldots \ldots \quad (1a)$$

in reversible processes; i.e. when a small quantity $d'Q$ of heat is added, S increases by the amount $d'Q/T$.

Since we have identically

$$T\,dS = d(TS) - S\,dT,$$

(1) can be written

$$d(U - TS) = -S\,dT + \frac{1}{4\pi}\mathbf{E}\,d\mathbf{D}. \quad \ldots \quad (1b)$$

If we now charge the condenser *isothermally* ($dT = 0$), the *free energy*

$$F = U - TS$$

is changed by the amount $\mathbf{E}\,d\mathbf{D}/4\pi$. If $\mathbf{D} = K\mathbf{E}$, we have therefore

$$F = \frac{1}{4\pi}\int_0^{\mathbf{E}} \mathbf{E}\,d\mathbf{D} = \frac{K}{8\pi}\mathbf{E}^2, \quad \ldots \quad (1c)$$

where K can now be an arbitrary function of the temperature.

If, generalizing (1), we denote by $d'A$ the work done on the system in a small change, then by the first law we have

$$dU = d'Q + d'A. \quad \ldots \ldots \quad (1d)$$

For reversible processes, by the second law, $d'Q = T\,dS$, and therefore

$$d(U - TS) = -S\,dT + d'A. \quad \ldots \quad (1e)$$

Accordingly, for adiabatic processes $d'A = dU$, but for isothermal processes $d'A = dF$. So far as this goes, the free energy plays the same kind of part in isothermal reversible processes as the energy itself does in adiabatic processes.

We have therefore justified our previous statement of account, but it has to be recognized that the balance arrived at is not a balance of the energy itself, but only a balance of the free energy, and that the general principle in question is the second law of thermodynamics, not the first.

That being understood, we are now faced with the problem of actually estimating the quantities of heat involved in cases of electric and magnetic polarization. In particular, we have still to determine how the energy U changes in our isothermal processes, now that we know that our statements up to this point are to be taken as referring to the free energy only.

2. Thermal Effects at Constant Volume.

We shall apply the fundamental thermodynamical equations (1d) and (1e) to a cubic centimetre of a substance, whose energy can be changed, first by the addition of a quantity of heat $d'Q$, and secondly by the performance of work, electrical or magnetic. We assume that the specific volume of the substance is not thereby altered to any sensible extent. Further, we confine ourselves to the case when the vectors $\mathbf{E}$ and $\mathbf{D}$ are parallel to one another at every point, and similarly $\mathbf{H}$ and $\mathbf{B}$, so that from the start we can deal with the numerical values E, D, H, B. By the first law (1d) we have then

$$dU = d'Q + \frac{1}{4\pi} E\,dD \text{ (in the electrical case)},$$

and $$dU = d'Q + \frac{1}{4\pi} H\,dB \text{ (for magnetization)}.$$

It is easy to see from the elementary treatment of the plate condenser (p. 70), and of the coil wound on a ring (p. 127), that the quantities of work there written down represent actual concrete amounts of energy, measurable e.g. in kilowatt-hours, and that no assumptions of any sort are made as to the form of the functions $B = B(H)$ and $D = D(E)$.

In what follows we confine ourselves to the case of *electric* polarization, seeing that the formulæ for the magnetic case can be deduced from the others by simply altering the symbols, i.e. by changing E, D, K into H, B, μ.

We also introduce, instead of D, the electric polarization P, by means of the equation

$$D = E + 4\pi P;$$

we then have $$dU = d'Q + d(E^2/8\pi) + E\,dP.$$

In order to simplify the notation, we shall split up the total energy into a "vacuum part" $E^2/8\pi$ and a part U' belonging to the material of the dielectric:

$$U = E^2/8\pi + U'. \qquad \qquad (2a)$$

It follows that $$dU' = T\,dS + E\,dP. \qquad \qquad (2b)$$

We shall not assume to begin with that P is directly proportional to the field strength E, but shall suppose instead that the polarization

$$P = P(E, T)$$

has been determined by experiment for the substance in question, so that P is known as a function of E and T. We have now to determine what these data enable us to say about the function

$$U' = U'(E, T).$$

If we solve (2*b*) for dS, and regard U' and P as expressed in terms of E and T, we find

$$dS = \frac{1}{T}\left(\frac{\partial U'}{\partial T} - E\frac{\partial P}{\partial T}\right)dT + \frac{1}{T}\left(\frac{\partial U'}{\partial E} - E\frac{\partial P}{\partial E}\right)dE. \quad . \quad (2c)$$

In order that the right-hand side may be the complete differential of the function $S = S(E, T)$, we must have the identity

$$\frac{\partial}{\partial E}\left(\frac{\partial S}{\partial T}\right) = \frac{\partial}{\partial T}\left(\frac{\partial S}{\partial E}\right),$$

where $\partial S/\partial T$ and $\partial S/\partial E$ are given by the factors of dT and dE on the right side of the preceding equation. On working out this " condition of integrability ", we find at once

$$\frac{\partial U'}{\partial E} - E\frac{\partial P}{\partial E} = T\frac{\partial P}{\partial T}. \quad . \quad . \quad . \quad . \quad . \quad (2d)$$

Since we are regarding the function $P(E, T)$ as empirically given, this equation gives the information we wanted about the change in U' due to a change in E under isothermal conditions.

If, as a special case, the function P is linear in E, i.e. if

$$P(E, T) = \chi(T)E, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2e)$$

the ratio $P : E$ for the substance, though still depending on T, will be independent of E. This is the case in most fluids. We shall then have, by (2*d*),

$$\frac{1}{E}\frac{\partial U'}{\partial E} = \chi + T\frac{d\chi}{dT} = \frac{d(\chi T)}{dT}. \quad . \quad . \quad . \quad . \quad . \quad (2f)$$

Here the right side is independent of E.

Since
$$\frac{1}{E}\frac{\partial U'}{\partial E} = 2\frac{\partial U'}{\partial E^2},$$

(2*f*) gives
$$U'(E, T) = f(T) + \tfrac{1}{2}E^2\left(\chi + T\frac{d\chi}{dT}\right), \quad . \quad . \quad . \quad . \quad (2g)$$

where f, a function of the temperature alone, is entirely unknown to begin with, and in particular involves the specific heat of the substance.

By (2*a*) we now obtain for the total energy density

$$U = f(T) + \frac{1}{8\pi}E^2\left\{1 + 4\pi\chi + T\frac{d(4\pi\chi)}{dT}\right\},$$

or, introducing the dielectric constant

$$K = 1 + 4\pi\chi,$$

$$U = f(T) + \frac{K}{8\pi}E^2\left(1 + \frac{T}{K}\frac{dK}{dT}\right). \quad . \quad . \quad . \quad (2h)$$

So long as we only consider isothermal changes, the function of the temperature, $f(T)$, is of no significance. In that case, therefore, the energy density bears to the free energy $KE^2/8\pi$ the ratio

$$1 + \frac{T}{K}\frac{dK}{dT} : 1.$$

Hence *it is only when the dielectric constant is independent of the temperature that the energy density can be taken as simply* $KE^2/8\pi$.

In many substances the ratio $P : E$ (the "susceptibility") is inversely proportional to T, or $\chi(T) = C/T$, so that $T\chi(T)$ is independent of the temperature. In this case, by (2*f*), U' is independent of E; and the electrical part of the energy density, viz. $U - f(T)$, is, not $KE^2/8\pi$, but simply $E^2/8\pi$.

We shall now investigate the quantity of heat $d'Q = T\,dS$ which must be supplied to the substance in order to keep its temperature constant when E changes. The result comes at once from (2*c*), by putting $dT = 0$ and taking account of (2*d*); thus

$$d'Q = T\left(\frac{\partial P}{\partial T}\right)_E dE. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2i)$$

Hence, in those cases for which (2*e*) holds,

$$d'Q = \tfrac{1}{2}T\frac{d\chi}{dT}\,d(E^2).$$

The quantity of heat absorbed per unit volume, while the intensity of the field increases from 0 to E, is therefore

$$Q = \tfrac{1}{2}E^2T\frac{d\chi}{dT}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2k)$$

Hence, if χ diminishes as the temperature increases, electric polarization is associated with negative absorption, i.e. the substance gives up heat to the surroundings.

The Specific Heats.—We can heat the dielectric either at constant polarization P (i.e. without change of its electrical state), or at constant field strength E. The latter case is the simpler to realize experimentally, since all that is necessary for the purpose is to keep the difference of potential of the condenser plates constant. The difference of the corresponding specific heats γ_P and γ_E can be determined completely, whenever the function $P(E, T)$ is known. To calculate γ_P, regard U' in (2*b*), p. 234, as a function of P and T. We then have

$$d'Q = T\,dS = \left(\frac{\partial U'}{\partial T}\right)_P dT + \left\{\left(\frac{\partial U'}{\partial P}\right)_T - E\right\} dP. \quad . \quad (2l)$$

For $dP = 0$, we have therefore

$$\gamma_P = \left(\frac{T\,\partial S}{\partial T}\right)_P = \left(\frac{\partial U'}{\partial T}\right)_P.$$

On the other hand, for $dE = 0$ we have from (2c)

$$\gamma_E = \left(\frac{\partial U'}{\partial T}\right)_E - E\left(\frac{\partial P}{\partial T}\right)_E.$$

But if we regard the value of P (as a function of E and T) as inserted in the equation $U' = U'(P, T)$, we have

$$\left(\frac{\partial U'}{\partial T}\right)_E = \left(\frac{\partial U'}{\partial T}\right)_P + \left(\frac{\partial U'}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_E.$$

It follows that $$\gamma_E = \gamma_P + \left(\frac{\partial P}{\partial T}\right)_E \left\{\left(\frac{\partial U'}{\partial P}\right)_T - E\right\}.$$

Also, from (2l), by the condition of integrability,

$$\left(\frac{\partial U'}{\partial P}\right)_T - E = -T\left(\frac{\partial E}{\partial T}\right)_P.$$

Hence $$\gamma_E = \gamma_P - T\left(\frac{\partial P}{\partial T}\right)_E \left(\frac{\partial E}{\partial T}\right)_P. \quad \quad (2m)$$

The last differential coefficient here is found from the function $P(E, T)$, which we are regarding as given:

$$\left(\frac{\partial E}{\partial T}\right)_P = -\left(\frac{\partial P}{\partial T}\right) \Big/ \left(\frac{\partial P}{\partial E}\right).$$

Hence, finally $$\gamma_E = \gamma_P + T\left(\frac{\partial P}{\partial T}\right)^2 \Big/ \left(\frac{\partial P}{\partial E}\right). \quad \quad (2n)$$

If in particular we put $$P = \chi(T)E,$$

then $$\gamma_E = \gamma_P + \frac{T}{\chi}\left(\frac{d\chi}{dT}\right)^2 E^2.$$

If—to specialize still further—χ is inversely proportional to the absolute temperature, or $\chi = C/T$, then

$$\gamma_E = \gamma_P + \frac{\chi}{T} E^2.$$

3. Thermodynamical Theory of Electrostriction.

In this section we shall apply the first and second laws of thermodynamics to the arrangement shown in fig. 2, p. 102, viz. two plates A and B of a condenser connected with the poles of a battery, so that there is a definite homogeneous field between them of strength E. The plates are partially immersed in a liquid dielectric, the boundary of which—

apart from mere rigid walls—consists of two pistons; one of these, fitted between the plates, is under a pressure p'; and the other, which is outside the field, under a pressure p_0. The whole system is in thermal contact with a large reservoir of heat at the absolute temperature T.

The four quantities mentioned, E, p', p_0, and T, cannot, it is clear, be arbitrarily assigned independently of one another, if there is equilibrium. It is intuitively evident, in fact, that three of them, say E, p_0, and T, can be given arbitrary values, but that there is only one definite value of p' compatible with these, if the whole of the liquid is not to be driven into or out of the condenser. *Hence there must be one relation, and one only, between the above four quantities.* The object of this section is to find this relation.

We shall use the following notation. For the part of the dielectric between the condenser plates, let V_1 be the total volume, v_1 the specific volume, m_1 the mass. For the part of the dielectric outside the field, let the corresponding quantities be denoted by V_0, v_0, m_0. Let U, S, M be the energy, entropy, and mass of the whole dielectric. Then obviously we have

$$V_1 = m_1 v_1;\ V_0 = m_0 v_0;\ M = m_1 + m_0. \quad . \quad . \quad (3a)$$

We suppose the physical nature of the dielectric to be defined by an *electrical* and a *thermal* equation of state; that is to say, the polarization P is a given function of the field strength, specific volume, and temperature; and similarly the pressure p in the dielectric, when there is no field, is a given function of the specific volume and the temperature. We therefore know the two functions

$$P = P(E, v_1, T) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3b)$$

and

$$p = p(v, T). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3c)$$

For a reversible change of U the general equation (1d), p. 233, becomes in the present case

$$dU = T\,dS + E\,d(V_1 P) - p'\,dV_1 - p_0\,dV_0. \quad . \quad (3d)$$

Now, for an *isolated* system, the condition of equilibrium is that the entropy has its greatest possible value. If then S is the entropy of our dielectric, S' that of other systems associated with ours (i.e. the heat reservoir, the electric battery, and the apparatus for maintaining the pressures p' and p_0), then we have equilibrium when, for given values of T, E, p', p_0, the total entropy $S + S'$ is a maximum with respect to flow of the liquid into or out of the condenser. Such flow implies changes in the masses m_1 and m_2, subject to the fixed relation $m_1 + m_2 = M$. The condition of equilibrium is therefore that

$$\delta S + \delta S' = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3e)$$

when $\delta m_1 = -\delta m_2$, and $\delta T = \delta E = \delta p' = \delta p_0 = 0$. But, if we

consider the external part of the system alone, the entropy of which is S', we see that

$$T\,\delta S' = -\delta U + E\,\delta(V_1 P) - p'\,\delta V_1 - p_0\,\delta V_0.$$

For, in this part of the system, the change of energy, as also the separate items of work, have always in a given process the opposite signs to the corresponding quantities in the dielectric.

If we multiply (3e) by $(-T)$, and substitute the expression just found for $T\,dS'$, noting that T, E, p', p_0 can be taken within the sign δ, since $\delta T = \delta E = \delta p' = \delta p_0 = 0$, we obtain, with reference to the variation indicated in (3e),

$$\delta(U - TS - EV_1P + p'V_1 + p_0V_0) = 0. \qquad (3f)$$

The quantity whose variation is taken in (3f) is called the thermodynamic potential ψ. By (3d), for an arbitrary change of ψ, with reversible changes of T, E, p', p_0, we have

$$d\psi = -S\,dT - V_1P\,dE + V_1\,dp' + V_0\,dp_0. \qquad (3g)$$

On the other hand, we can write S in the form

$$S = m_1 s_1 + m_0 s_0,$$

where s_1 and s_0 denote the entropies per unit mass in the field and outside it; then, using (3a), we find

$$d\psi = m_1(-s_1\,dT - v_1P\,dE + v_1\,dp') + m_0(-s_0\,dT + v_0\,dp_0).$$

Hence, introducing the specific potentials, i.e. the potentials per unit mass, we have

$$\left.\begin{aligned} &d\psi_1 = -s_1\,dT - v_1P\,dE + v_1\,dp';\ \psi_1 = \psi_1(T, E, p') \\ \text{and}\quad &d\psi_0 = -s_0\,dT + v_0\,dp_0;\ \psi_0 = \psi_0(T, p_0). \end{aligned}\right\} \qquad (3h)$$

The thermodynamical potential

$$\psi \equiv \psi(m_1, m_0, T, E, p', p_0) = m_1\psi_1 + m_0\psi_0$$

may be regarded as a *function of the six variables* m_1, m_0, T, E, p', and p_0. If the total differential of this function of six variables is to be consistent with the formula (3g), which holds for equilibrium processes, we must have

$$\frac{\partial\psi}{\partial m_1}\,\delta m_1 + \frac{\partial\psi}{\partial m_2}\,\delta m_2 = 0,$$

whenever $\qquad \delta m_1 + \delta m_2 = 0.$

Hence $\qquad \dfrac{\partial\psi}{\partial m_1} = \dfrac{\partial\psi}{\partial m_2}$, or (since $\psi = m_1\psi_1 + m_0\psi_0$)

$$\psi_1 = \psi_0,$$

i.e.

$$\psi_1(T, E, p') = \psi_0(T, p_0). \qquad (3i)$$

This equation gives the required general relation between the four variables T, E, p′, *and* p_0.

We shall discuss (3*i*) for the special case of *isothermal* changes of E, p', p_0; i.e. we regard T as having a given fixed value, and study the relationship between E, p', and p_0 which is defined by (3*i*). In differential form it becomes

$$\frac{\partial \psi_1}{\partial E}\,dE + \frac{\partial \psi_1}{\partial p'}\,dp' = \frac{\partial \psi_0}{\partial p_0}\,dp_0,$$

or, if we insert from (3*h*) the values of the partial derivatives,

$$-v_1 P\,dE + v_1\,dp' = v_0\,dp_0. \quad . \quad . \quad . \quad . \quad (3k)$$

This simple equation summarizes the facts about electrostriction (§ 6, p. 95) *and also about electrical surface forces* (§ 7, p. 100). Note in the first place that the right-hand term in (3*k*), in virtue of the thermal equation of state (3*c*), can be regarded as the complete differential of the function

$$f(v_0) = \int_a^{v_0} v\,\frac{\partial p}{\partial v}\,dv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3l)$$

On the other hand, p' must be uniquely determinate when v_1 and E are given, so that we have

$$dp' = \frac{\partial p'}{\partial v_1}\,dv_1 + \frac{\partial p'}{\partial E}\,dE,$$

and therefore, from (3*k*),

$$\left[-v_1 P + v_1\frac{\partial p'}{\partial E}\right] dE + v_1\frac{\partial p'}{\partial v_1}\,dv_1 = df(v_0). \quad . \quad (3m)$$

Here the left side, like the right, must be a complete differential.

Hence
$$\frac{\partial}{\partial v_1}\left(-v_1 P + v_1\frac{\partial p'}{\partial E}\right)_E = \frac{\partial}{\partial E}\left(v_1\frac{\partial p'}{\partial v_1}\right)_{v_1},$$

i.e.
$$\frac{\partial p'}{\partial E} = \frac{\partial}{\partial v_1}(v_1 P). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3n)$$

We shall now suppose that the electrical equation of state (3*b*) takes the special form

$$P = \chi(T, v_1)E. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3o)$$

Then (3*n*) becomes
$$\frac{\partial p'}{\partial (E^2)} = \tfrac{1}{2}\,\frac{\partial (v_1\chi)}{\partial v_1},$$

or, on integration,
$$p' = p(v_1) + \tfrac{1}{2}E^2\,\frac{\partial (v_1\chi)}{\partial v_1}, \quad . \quad . \quad . \quad . \quad (3p)$$

which agrees exactly with the results of the electrodynamical treat-

ment of § 7, p. 102. Here $p(v_1)$ is the pressure which the dielectric, in accordance with the equation of state (3c), would exert on the walls of the containing vessel, at specific volume v_1, and with no electric field.

If in (3m) we insert the values (3o) for P and (3p) for p' we obtain

$$\tfrac{1}{2}v_1\left\{-\chi+\frac{\partial(v_1\chi)}{\partial v_1}\right\}d(E^2)+\left\{v_1\frac{\partial p(v_1)}{\partial v_1}+\tfrac{1}{2}E^2v_1\frac{\partial^2(v_1\chi)}{\partial v_1^2}\right\}dv_1=df(v_0),$$

or, by (3l),
$$d\left(\tfrac{1}{2}v_1^2\frac{\partial\chi}{\partial v_1}E^2\right)+df(v_1)=df(v_0).$$

Hence (3k) can be completely integrated for the special case (3o). Clearly we have

$$f(v_1)-f(v_0)=\int_{v_0}^{v_1}v\frac{\partial p}{\partial v}\,dv,$$

so that the integral is

$$\int_{v_0}^{v_1}v\frac{\partial p}{\partial v}\,dv=-\tfrac{1}{2}E^2v_1^2\frac{\partial\chi}{\partial v_1}, \qquad . \quad . \quad . \quad . \quad (3q)$$

which is exactly equivalent to the formula (14c) for electrostriction already proved and discussed in detail (p. 97). There, however, instead of the specific volume v we used the reciprocal $1/\sigma$ of the density, and instead of the susceptibility χ the dielectric constant K ($=1+4\pi\chi$).

CHAPTER XII

The Forces in Fields which Vary with the Time

1. The Maxwell Stresses and the Principle of Action and Reaction.

In the discussions of § 5 (p. 91) and § 2 (p. 146) on mechanical forces we confined ourselves to bodies at rest in a steady electromagnetic field. In § 8 (p. 104) and § 2 (p. 146) we succeeded in representing the mutual action between two systems of charges and material bodies, by means of forces acting across any surface separating the too systems. For these forces, action and reaction are certainly equal, as they are required to be by Newton's third law; in fact, when the direction of the normal to an element of surface is reversed, the surface traction derived from the stress tensor changes its sign.

The question arises: what happens with regard to these forces in fields which vary rapidly with the time? The Maxwell-Hertzian theory assumes that the whole force exerted on a bounded region can still be represented, even in an electromagnetic field varying as rapidly as we please, by the surface tractions derived from the Maxwellian stress tensor, so that the force may be expressed in the form

$$\mathbf{F} = \iint (\mathbf{T}_{\mathrm{el}} + \mathbf{T}_{\mathrm{mag}})\, dS. \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

If n is the outward normal to dS, the x-components of $\mathbf{T}_{\mathrm{el}}$ and $\mathbf{T}_{\mathrm{mag}}$ are given by

$$\left.\begin{aligned} (T_{\mathrm{el}})_x &= \frac{K}{4\pi}(E_x^2 - E_y^2 - E_z^2)\cos(n, x) \\ &\qquad + \frac{K}{4\pi}E_xE_y\cos(n, y) + \frac{K}{4\pi}E_xE_z\cos(n, z); \\ (T_{\mathrm{mag}})_x &= \frac{\mu}{4\pi}(H_x^2 - H_y^2 - H_z^2)\cos(n, x) \\ &\qquad + \frac{\mu}{4\pi}H_xH_y\cos(n, y) + \frac{\mu}{4\pi}H_xH_z\cos(n, z). \end{aligned}\right\} \quad (1b)$$

We shall now, for the case of fields varying rapidly with the

time, investigate the value of the force $\mathbf{f}$ per unit volume, where

$$\mathbf{F} = \int \mathbf{f}\, dV. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1c)$$

In the first place, we find by means of (9) and (9*a*), p. 152, that

$$\mathbf{f} = \frac{1}{4\pi}\mathbf{E}\operatorname{div}(K\mathbf{E}) - \frac{1}{8\pi}\mathbf{E}^2 \operatorname{grad} K + \frac{1}{4\pi}[\operatorname{curl}\mathbf{E}, K\mathbf{E}]$$
$$+ \frac{1}{4\pi}\mathbf{H}\operatorname{div}(\mu\mathbf{H}) - \frac{1}{8\pi}\mathbf{H}^2 \operatorname{grad}\mu + \frac{1}{4\pi}[\operatorname{curl}\mathbf{H}, \mu\mathbf{H}].$$

This expression for the force density is obviously composed of three distinct parts. In fact, if we take account of Maxwell's equations, viz.

$$\operatorname{div}(K\mathbf{E}) = 4\pi\rho; \quad \operatorname{div}(\mu\mathbf{H}) = 0;$$
$$\operatorname{curl}\mathbf{E} = -\frac{\mu}{c}\dot{\mathbf{H}}; \quad \operatorname{curl}\mathbf{H} = \frac{4\pi}{c}\mathbf{i} + \frac{K}{c}\dot{\mathbf{E}};$$

we find
$$\mathbf{f} = \mathbf{f}_{\mathrm{el}} + \mathbf{f}_{\mathrm{mag}} + \mathbf{f}_N. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1d)$$

where
$$\mathbf{f}_{\mathrm{el}} = \rho\mathbf{E} - \frac{1}{8\pi}\mathbf{E}^2 \operatorname{grad} K, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1e)$$

$$\mathbf{f}_{\mathrm{mag}} = \frac{1}{c}[\mathbf{i}\,\mathbf{B}] - \frac{1}{8\pi}\mathbf{H}^2 \operatorname{grad}\mu, \quad . \quad . \quad . \quad . \quad . \quad (1f)$$

$$\mathbf{f}_N = \frac{K\mu}{4\pi c}\frac{\partial}{\partial t}[\mathbf{E}\mathbf{H}]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1g)$$

Of these three component parts of the force density the two first have already come before our notice (pp. 95, 150). They are zero, except at places where there are either charges or matter; we are therefore entitled to regard the sum $\mathbf{f}_{\mathrm{el}} + \mathbf{f}_{\mathrm{mag}}$ as the actual density of the force acting on the matter in the field—and that is what we did before. The third part of (1*d*), however, is new. It occurs only in fields which vary with the time. It is connected in the closest way with the Poynting vector representing the stream of energy (p. 145), viz.

$$\mathbf{N} = \frac{c}{4\pi}[\mathbf{E}\mathbf{H}].$$

We have in fact
$$\mathbf{f}_N = \frac{K\mu}{c^2}\frac{\partial \mathbf{N}}{\partial t}.$$

The remarkable thing about this force is that its existence does not presuppose the presence of matter; the Maxwellian stresses provide a "moving" force at points of empty space, even though there is no matter there for the force to move.

This result is of fundamental importance for the further development of the theory. Originally—i.e. in the time of Maxwell and Hertz

$-\mathbf{f}_N$ was interpreted as an actual force acting upon the "æther". At one time, when the tendency in all cases was to endow the æther in the most liberal way with mechanical properties, this interpretation was undoubtedly appropriate enough. Newton's third law is satisfied by the expression taken for the force in (1*a*); *nevertheless the law of action and reaction is no longer true for matter by itself alone.*

The reaction of the æther on matter is immediately accessible to observation in the form of radiation pressure. Imagine a finite train of waves originating at a source of light to be incident on a mirror. Further, let the source be so far away that the wave train has completely left it by the time the front of the wave reaches the mirror. The mirror will then be subjected to radiation pressure, which is practically the force defined by $\mathbf{f}_N$—but the light source is by that time entirely out of action. We have therefore here a force actually exerted by vacuous space on the mirror, and—since action and reaction are equal—an equal and opposite force exerted by the mirror on vacuous space.

The conception of a substantial æther, which may be acted upon by force, is incompatible with the views held at the present day. The modern theories (*Relativity*, *Theory of Electrons*) developed in the sixth volume of this series take no account of any forces beyond those which act on matter. The expression for the force density $\mathbf{f}$ to which these recent theories lead is accordingly different from that of (1*d*). The difference simply amounts to leaving out that part of $\mathbf{f}$ which, as we have just seen, implies the existence of a force acting on vacuous space. The force density obtained by the theory of relativity is

$$\mathbf{f}_{\text{mat}} = \mathbf{f}_{\text{el}} + \mathbf{f}_{\text{mag}} + \frac{K\mu}{c^2}\frac{\partial \mathbf{N}}{\partial t} - \frac{1}{c^2}\frac{\partial \mathbf{N}}{\partial t}. \quad . \quad . \quad . \quad (2)$$

(The terms additional to $\mathbf{f}_{\text{el}} + \mathbf{f}_{\text{mag}}$, which still appear on the right, vanish in a vacuum. In an experimental determination of $\mathbf{f}_{\text{mat}}$ they might, on account of their smallness, be omitted altogether.) The resultant force $\mathbf{F}_{\text{mat}}$ which, according to (2), acts on a finite volume, is therefore

$$\mathbf{F}_{\text{mat}} = \int(\mathbf{T}_{\text{el}} + \mathbf{T}_{\text{mag}})\, dS - \frac{d}{dt}\int \frac{1}{c^2}\mathbf{N}\, dV. \quad . \quad . \quad (2a)$$

If we denote by $\mathbf{G}_{\text{mat}}$ the momentum of the matter contained in the volume considered, and by $\mathbf{g}_{\text{mat}}$ the momentum density, we have

$$\mathbf{G}_{\text{mat}} = \int \mathbf{g}_{\text{mat}}\, dV.$$

The significance of $\mathbf{F}_{\text{mat}}$ is due to the fact that it defines the time rate of increase of $\mathbf{G}_{\text{mat}}$:

$$\mathbf{F}_{\text{mat}} = \frac{d}{dt}\mathbf{G}_{\text{mat}}.$$

We have therefore from (2a)

$$\int (\mathbf{T}_{\mathrm{el}} + \mathbf{T}_{\mathrm{mag}})\, dS = \frac{d}{dt}\int \left(\mathbf{g}_{\mathrm{mat}} + \frac{1}{c^2}\mathbf{N}\right) dV.$$

We shall now apply this equation to a bounded system, i.e. a system whose electromagnetic field lies entirely at a finite distance. In this case we may take the volume through which we integrate so large that the field vanishes all over its external boundary; hence we obtain

$$\int \mathbf{g}_{\mathrm{mat}}\, dV + \int \frac{1}{c^2}\mathbf{N}\, dV = \text{const.} \quad . \quad . \quad . \quad (2b)$$

But it follows from Newton's third law that the momentum of an isolated system is constant with respect to the time. Hence, if we wish to adhere to the third law, we are compelled by equation (2b) to regard the quantity

$$\mathbf{g}_{\mathrm{mat}} + \frac{1}{c^2}\mathbf{N}$$

as the density of momentum. We have here an example of the important result of the Theory of Relativity, that every stream of energy $\mathbf{N}$ has associated with it a momentum density

$$\mathbf{g}_N = \frac{1}{c^2}\mathbf{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(equivalence of energy and inertial mass).

In particular, electromagnetic radiation always carries with it the momentum specified in (3). We call $\mathbf{g}_N$ the momentum density of the radiation, and

$$\mathbf{G}_N = \int \mathbf{g}_N\, dV$$

we call the total momentum of the radiation contained in V. The theorem of momentum (2b) now takes the form:

The sum $$\mathbf{G}_{\mathrm{mat}} + \mathbf{G}_N$$

of the momentum of matter and the momentum of radiation in an isolated system is constant with respect to the time.

Page

Divergence: $\operatorname{div} \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$. 18

Gauss's theorem: $\iint v_n \, dS = \iiint \operatorname{div} \mathbf{v} \, dV$. 18

Green's theorem:

$$\iint \psi \frac{\partial \phi}{\partial n} dS = \iiint \{\psi \, \Delta\phi + (\operatorname{grad} \psi, \operatorname{grad} \phi)\} \, dV, \qquad 19$$

$$\iint \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}\right) dS = \iiint (\psi \, \Delta\phi - \phi \, \Delta\psi) \, dV. \qquad 19$$

Potential of point sources of strengths e_i:

$$\phi = \Sigma \frac{e_i}{r_i}; \ \mathbf{v} = -\operatorname{grad} \phi. \qquad 20$$

Double source of moment $\mathbf{m}$:

$$\phi = -\left(\mathbf{m}, \operatorname{grad}_r \frac{1}{r}\right) = \frac{(\mathbf{mr})}{r^3} = \frac{|\mathbf{m}| \cdot \cos\theta}{r^2}. \qquad 23$$

Surface of discontinuity:

$$\phi = \iint \frac{\omega}{r} dS_{12} - \iint \tau \left(\mathbf{n}, \operatorname{grad}_r \frac{1}{r}\right) dS_{12}. \qquad 29$$

Uniform double stratum: $\mathbf{v} = \tau \oint \frac{[d\mathbf{s}, \mathbf{r}]}{r^3}$. 32

Curl:

$$\operatorname{curl} \mathbf{v} = \mathbf{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right). \qquad 33$$

Stokes's theorem: $\oint \mathbf{v} \, d\mathbf{s} = \iint (\operatorname{curl} \mathbf{v})_n \, dS$. 35

Expression for $\mathbf{v}$ in terms of its sources ρ and vortices $\mathbf{c}$:

$$\mathbf{v} = -\operatorname{grad} \iiint \frac{\rho \, dV}{r} + \operatorname{curl} \iiint \frac{\mathbf{c} \, dV}{r}. \qquad 38$$

Rules for calculation with vectors:

$$\operatorname{div} \phi\mathbf{A} = \phi \operatorname{div} \mathbf{A} + \mathbf{A} \operatorname{grad} \phi, \qquad 18$$

$$\operatorname{curl} \phi\mathbf{A} = \phi \operatorname{curl} \mathbf{A} + [\operatorname{grad} \phi, \mathbf{A}], \qquad 129$$

$$\operatorname{div} [\mathbf{AB}] = \mathbf{B} \operatorname{curl} \mathbf{A} - \mathbf{A} \operatorname{curl} \mathbf{B}, \qquad 36$$

$$\operatorname{curl} \operatorname{curl} \mathbf{A} = \operatorname{grad} \operatorname{div} \mathbf{A} - \Delta\mathbf{A}. \qquad 36$$

ELECTRODYNAMICS

Page

Maxwell's equations for bodies at rest; I to IV, general; V to VII, constitutive: 144

I. $$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c}\mathbf{i} + \frac{1}{c}\dot{\mathbf{D}}.$$ 144

II. $$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\dot{\mathbf{B}}.$$ 141

III. $$\operatorname{div} \mathbf{D} = 4\pi\rho.$$ 75

IV. $$\operatorname{div} \mathbf{B} = 0.$$ 136

Here $\mathbf{H}$ = magnetic field strength,
$\mathbf{E}$ = electric field strength,
$\mathbf{D}$ = dielectric displacement,
$\mathbf{B}$ = magnetic induction,
$\mathbf{i}$ = current density,
ρ = density of (true) charge.

We have also the vectors

$\mathbf{P}$ = dielectric polarization,
$\mathbf{I}$ = intensity of magnetization,

with the relations

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P},$$ 74

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I}.$$ 136

Hence, from III,

$$\operatorname{div} \mathbf{E} = 4\pi(\rho - \operatorname{div} \mathbf{P}).$$

For isotropic substances,

V. $$\mathbf{i} = \sigma(\mathbf{E} + \mathbf{E}^{(e)}).$$ 116

VI. $$\mathbf{P} = \chi\mathbf{E};\quad \mathbf{D} = K\mathbf{E};$$ 74

where σ = electric conductivity,
$\mathbf{E}^{(e)}$ = impressed electromotive force,
χ = dielectric susceptibility,
K = dielectric constant ($= 1 + 4\pi\chi$).

For substances not ferromagnetic,

VII. $$\mathbf{I} = \kappa\mathbf{H};\quad \mathbf{B} = \mu\mathbf{H};$$

where κ = magnetic susceptibility,
μ = permeability ($= 1 + 4\pi\kappa$).

Page

Equation of Energy.—From I to VII it follows that

$$-\frac{d}{dt}\,\frac{1}{8\pi}\,(K\mathbf{E}^2+\mu\mathbf{H}^2)=\operatorname{div}\mathbf{N}+(\mathbf{iE}),$$ 145

where $\mathbf{N}$ is the *Poynting vector* $\frac{c}{4\pi}\,[\mathbf{EH}]$. 145

Energy density of the field:

$$u=\frac{1}{8\pi}\,(K\mathbf{E}^2+\mu\mathbf{H}^2).$$ 145

Thermochemical activity:

$$(\mathbf{iE})=\frac{1}{\sigma}\,\mathbf{i}^2-(\mathbf{iE}^{(e)}).$$ 150

Wave Equation.—For homogeneous uncharged media (K, μ, σ constant in space; $\mathbf{E}^{(e)}=0$; $\rho=0$):

$$\frac{K\mu}{c^2}\,\frac{\partial^2\mathbf{E}}{\partial t^2}+\frac{4\pi\sigma\mu}{c^2}\,\frac{\partial\mathbf{E}}{\partial t}=\Delta\mathbf{E}.$$ 188

Hence, for insulators:

Velocity of wave propagation $a=\frac{c}{\sqrt{(K\mu)}}$, 184

Index of refraction $n=\sqrt{(K\mu)}$; 185

and, for conductors:

Depth of penetration $d=\frac{1}{\sqrt{\mu}}\sqrt{\frac{\lambda_0 c}{\sigma}}$. 190

FURTHER SPECIALIZATIONS OF I TO VII

Quasi-steady fields: neglect of $\dot{\mathbf{D}}/c$ in I; equivalent to the assumption $\operatorname{div}\mathbf{i}=0$, and the assumption of infinitely great wave-velocity a. For a quasi-steady current $\mathbf{i}$ the *self-inductance* L is defined in terms of the magnetic field energy: 146

$$U_{\text{mag}}=\frac{1}{8\pi}\int\mu\mathbf{H}^2\,dV=\tfrac{1}{2}Li^2.$$ 164

Steady fields: $\dot{\mathbf{D}}=0$, $\dot{\mathbf{B}}=0$. 109

$\mathbf{E}$ is irrotational, so that a potential ϕ exists:

$$\mathbf{E}=-\operatorname{grad}\phi,$$ 109

$$\operatorname{curl}\mathbf{B}=\frac{4\pi}{c}\,(\mathbf{i}+c\operatorname{curl}\mathbf{I}).$$ 136

Page

Static fields: $\dot{\mathbf{D}} = 0,\ \dot{\mathbf{B}} = 0,\ \mathbf{i} = 0.$

The fields $\mathbf{E}$ and $\mathbf{H}$ are irrotational, and independent of one another:

magnetostatics: $\operatorname{div} \mathbf{H} = -4\pi \operatorname{div} \mathbf{I};$ 132

electrostatics: $\operatorname{div} (K\mathbf{E}) = 4\pi\rho.$ 75

Hence, in homogeneous insulators (K constant in space),

$$-\operatorname{div} \mathbf{E} = \Delta\phi = -\frac{4\pi}{K}\rho.$$ 75

NUMERICAL RELATIONSHIPS OF UNITS

	Name of Practical Unit	Measure in Electromagnetic Units	Measure in Electrostatic Units	Measure in Gaussian Units*
Electric charge	Coulomb	10^{-1}	3×10^9	3×10^9
Electromotive force (and electric potential)	Volt	10^8	$1/300$	$1/300$
Electric intensity ..	Volt/cm.	10^8	$1/300$	$1/300$
Capacity	Farad	10^{-9}	9×10^{11}	9×10^{11}
Current	Ampere	10^{-1}	3×10^9	3×10^9
Resistance	Ohm	10^9	$1/(9 \times 10^{11})$	$1/(9 \times 10^{11})$
Inductance	Henry	10^9	$1/(9 \times 10^{11})$	$1/(9 \times 10^{11})$
Magnetic pole strength		1	$1/(3 \times 10^{10})$	1
Magnetic intensity ..	Gauss	1	3×10^{10}	1
Magnetic flux	Maxwell	1	$1/(3 \times 10^{10})$	1
Energy (and work) ..	Joule	10^7 ergs	10^7 ergs	10^7 ergs
Power	Watt	10^7 ergs/sec.	10^7 ergs/sec.	10^7 ergs/sec.

The table shows the conversion factor required to change from any one to any other of the four systems of units. If, for example, a capacity is given in electrostatic units, and is to be expressed in electromagnetic units, we can use the relation stated in the table that 9×10^{11} e.s.u. $= 10^{-9}$ e.m.u., so that 1 e.s.u. $= 1/(9 \times 10^{20})$ e.m.u. A useful mnemonic is the fact that the equation

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dE}{dt}$$

holds in all four systems of units.

* I.e. electrical quantities in electrostatic units, magnetic quantities in electromagnetic units.

EXAMPLES

But be ye doers of the word, and not hearers only, deceiving your own selves.— James, i, 22.

Vectors

1. The length of the vector $\mathbf{A}$ is $|\mathbf{A}| = 5$; it makes an angle of 30° with the z-axis; its projection on the xy-plane makes an angle of 45° with the x-axis.
The projection of the vector $\mathbf{B}$ on the z-axis is $B_z = +4$; its projection on the xy-plane has the length 6, and makes with the x-axis an angle +120°.
Find $\mathbf{A} + \mathbf{B}$. Give the length of this vector, its Cartesian components, and the angles it makes with the co-ordinate axes.

2. The point of application of the force $\mathbf{P} = (5, 10, 15)$ kg. is displaced from the point (1, 0, 3) to the point (3, −1, −6). Find the work done by the force, in kg.-cm. (The co-ordinates are in cm.)

3. Find the scalar product of two diagonals of a unit cube. What is the angle between them?

4. The unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ of a Cartesian co-ordinate system define a cube of volume 1, of which they form the edges. Find the scalar and the vector product of two face-diagonals starting from the origin. What angle do they form with one another? What is the area of a face of a tetrahedron inscribed in the cube?

5. Find the Cartesian components of that vector, of length 3, which makes equal angles with the negative x-axis, the positive y-axis and the negative z-axis; the vector being drawn from the origin in the octant formed by the parts of the axes as stated.

6. Let $\mathbf{a}$ be a vector drawn from the origin, and $\mathbf{a}_0$ the corresponding unit vector; also let $\mathbf{r}$ be the vector drawn from the origin to the variable point P. Prove that

$$(\mathbf{a}_0 \cdot \mathbf{r}) = |\mathbf{a}|$$

is the equation of the plane at right angles to $\mathbf{a}$ through its end-point.

7. The perpendicular from the origin of co-ordinates to the plane P is $\mathbf{p} = (2, 4, 6)$. A straight line through the origin has the direction cosines $(1/\sqrt{2}, 1/\sqrt{2}, 0)$. Find the co ordinates of its point of intersection with the plane P.

8. Let $\mathbf{n}$ be a unit vector and $\mathbf{A}$ an arbitrary vector; prove the formula

$$\mathbf{A} = \mathbf{n}(\mathbf{A} \cdot \mathbf{n}) + [\mathbf{n} \cdot [\mathbf{A} \cdot \mathbf{n}]]$$

and show that it represents the decomposition of $\mathbf{A}$ into two components, one parallel to $\mathbf{n}$ and the other perpendicular to $\mathbf{n}$.

9. The vector (2, 2, 5) is drawn from the origin; find the vector perpendicular to this from the point (1, 2, 1).

10. Given the vectors $\mathbf{a} = (2, 1, 1)$ and $\mathbf{b} = (-1, 3, 2)$; find the components of the fundamental vectors of the Cartesian left-handed co-ordinate system, whose x-axis is in the direction of $\mathbf{a}$, while its y-axis is in the plane of $\mathbf{a}$ and $\mathbf{b}$ and on the same side of $\mathbf{a}$ as $\mathbf{b}$ is.

11. Find the components of the vector $\mathbf{r} = (5, 8, 10)$ in the co-ordinate system specified in the preceding example.

12. The end-points of the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ drawn from the origin of co-ordinates define a plane. What is its distance from the origin?

[Use the scalar products of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ with the (unit) normal to the plane, each of which is equal to the distance required; then solve the equations for the components of the unit normal.]

What is the geometrical interpretation of the final formula?

[Note that the volume of the tetrahedron formed by three adjacent edges of a parallelepiped is $\frac{1}{6}$ the volume of the parallelepiped.]

13. A body is rotating about a fixed axis; the angular velocity vector is $\mathbf{u} = (-50, +80, +100)$; the velocity $\mathbf{v}$ of the point $\mathbf{r} = (4, 5, 6)$ has the x-component $v_x = 20$, and the y-component $v_y = 30$. Find the shortest distance of the axis of rotation from the origin.

14. A rigid body rotates at 300 revolutions per minute about an axis which makes with the x-axis an angle of 50° and with the y-axis an angle of 70°, and passes through the origin in the first octant. Find the components of the velocity of the point $\mathbf{r} = (4, 5, 6)$ cm.

15. The end-points of three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ lying in a plane, and drawn from the origin of co-ordinates, determine a triangle. Find its area, having given $\mathbf{a} = (5, 1)$, $\mathbf{b} = (8, 6)$, $\mathbf{c} = (2, 10)$.

16. The vectors $\mathbf{a} = (6, 3, 1)$, $\mathbf{b} = (3, 6, 1)$, $\mathbf{c} = (1, 3, 6)$ start from the origin of co-ordinates. Find the area of the triangle formed by their end-points.

17. For any determinant

$$D = \begin{vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}, & a_{n2}, & \ldots, & a_{nn} \end{vmatrix} \qquad (a_{ij} \text{ real})$$

Hadamard has proved the inequality

$$\begin{aligned} D^2 &\leqslant \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}^2 \right) \\ &= (a_{11}^2 + a_{12}^2 + \ldots + a_{1n}^2)(a_{21}^2 + a_{22}^2 + \ldots + a_{2n}^2) \ldots (a_{n1}^2 + a_{n2}^2 + \ldots + a_{nn}^2). \end{aligned}$$

Consider the case of a three-rowed determinant, and regard the elements of a row as components of a vector; what geometrical theorem is then expressed by the inequality? What theorem of plane geometry corresponds to the case of a two-rowed determinant?

18. With every point of a curve in space there is associated a unit vector $\mathbf{t}$, the direction of which is that of the velocity of a point describing the curve in an assigned sense. This vector therefore gives at the same time the direction of the tangent. Prove that $\left(\mathbf{t}\,.\,\frac{d\mathbf{t}}{ds}\right) = 0$, where s is the length of the arc of the curve measured from a fixed point on it. What is the geometrical meaning of $\frac{d\mathbf{t}}{ds}$?

19. Let $\mathbf{r}$ be the radius vector from the origin to any point, and $\mathbf{a}$ a constant vector. Find the gradient of the scalar product of $\mathbf{a}$ and $\mathbf{r}$.

20. If φ and ψ are two functions of a point in space, so that $\varphi \equiv \varphi(x, y, z)$ and $\psi \equiv \psi(x, y, z)$, what is the geometrical signification of $[\operatorname{grad}\varphi\,.\,\operatorname{grad}\psi]$?

21. A circular disk rotates about a fixed point in its plane with angular velocity ω. Every point of the plane lying within the circle has thus a definite velocity vector $\mathbf{v}$ associated with it. Find an expression for curl $\mathbf{v}$.

22. The points of a plane rotate about a fixed point in the plane, but with angular velocities which depend on the distance from the fixed point: $\omega = \omega(r)$. Determine the function ω, if the velocity field is irrotational.

23. A plane central field $\mathbf{A}$ is defined by $\mathbf{A} = \mathbf{r}\,.\,f(|\,\mathbf{r}\,|) = \mathbf{r}\,.\,f(r)$. Determine $f(r)$ so that the field may be irrotational and solenoidal.

24. A central field $\mathbf{A}$ in space is defined by $\mathbf{A} = \mathbf{r}\,.\,f(|\,\mathbf{r}\,|) = \mathbf{r}\,.\,f(r)$. Determine $f(r)$ so that the field may be irrotational and solenoidal.

25. The rectangular components of a vector $\mathbf{A}$ are:

$$A_x = y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y},\quad A_y = z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z},\quad A_z = x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x},$$

where f is a given function of the co-ordinates x, y, z. Express $\mathbf{A}$ as the vector product of two vectors, and prove that

$$(\mathbf{A}\,.\,\mathbf{r}) = 0 \quad\text{and}\quad (\mathbf{A}\,.\,\operatorname{grad} f) = 0.$$

Electrostatics

26. Find the force of attraction between two equal and opposite charges of 1 coulomb distant 1 km. from each other.

27. A quantity of positive electricity $+e$ is distributed uniformly throughout the volume of a sphere of radius a. A negative point charge $-e$ is situated at a point within this "charge-cloud". Find the force acting on the point charge, as a function of the distance from the centre of the sphere.

28. A dipole $\mathbf{m}$ is situated at a certain point in a field of strength $\mathbf{E}$. Find the work required to remove the dipole to an infinite distance. (The angle between $\mathbf{E}$ and $\mathbf{m}$ is α.)

What does the result become in the special case when the dipole can rotate freely?

29. A point charge of 0·5 electrostatic units is placed at a distance of 1 cm. from the plane face of a large body of

(*a*) metal (conducting plane), or
(*b*) glass with dielectric constant 7 (dielectric half-space).

Find in each case the force which acts on the charge.

30. A metal sphere of radius a is placed in a homogeneous electric field $\mathbf{E}$. Find the surface density of the induced charge. Find also the surface density of the free charge on a dielectric sphere whose dielectric constant is K.

31. Find the force between a metal sphere of radius R carrying a charge E, and a small body with a charge e, at a distance d from the centre of the sphere. Prove that the two bodies in certain circumstances may attract each other, even if the charges E and e are of like sign.

32. Find the greatest charge which can be carried by a metal sphere of 10 cm. diameter, if the dielectric strength of the air is 20,000 volts/cm.

33. Find the limiting radius of curvature to which the corners of a conductor charged to 100,000 volts must be rounded, if the dielectric strength of the air is 20,000 volts/cm. (Take the rate of fall of potential in the immediate neighbourhood of the surface as approximately equal to that for a sphere.)

34. What is the surface density of electric charge at a place on the earth's surface where the rate of fall of potential is 250 volts/m.? Find the force acting on 1 sq. m. of the earth's surface at this place.

35. A soap bubble hanging from the end of a glass tube shrinks towards the opening of the tube under the action of surface tension. Is it possible, in view of the fact that air has a dielectric strength of 20,000 volts/cm., to prevent the soap bubble from collapsing completely by giving it a strong electric charge? If so, find its limiting diameter. (Surface tension = 48·5 dynes/cm.)

36. In a charged soap bubble of radius r the air pressure is the same inside and outside the bubble. Find the relation between the radius, potential and surface tension.

37. A very long thin rod of dielectric constant K is placed in a homogeneous field $\mathbf{E}_0$, parallel to the direction of the field. Find the values of $\mathbf{E}$ and $\mathbf{D}$ in the interior of the rod.

38. A homogeneous field of intensity $\mathbf{E}$ exists within a dielectric of which the dielectric constant is K. Find the intensity in a hollow space within the dielectric in the three cases when the hollow has the form of

(*a*) a very long thin cylinder parallel to the lines of force;
(*b*) a thin plate perpendicular to the direction of the field;
(*c*) a sphere (cf. p. 80).

39. In a plate condenser (plate distance d) which is connected to a battery of V volts, glass plates (dielectric constant K) of different thicknesses are inserted in turn. How do the two values of the field strength, viz. in the glass and in the residual air space, depend on the thickness x of the glass plate? How does the capacity of the condenser vary with x? Also find the values of the field strength in the glass and in the air space when the condenser is disconnected from the battery before the glass plate is inserted.

40. (*a*) With what force (per sq. cm.) do the coatings of a plate condenser attract each other, at 1000 volts pressure and 1 mm. plate distance?

(*b*) What is the value of the force when the condenser is disconnected from the battery of 1000 volts after charging, and is then filled up with petroleum ($K = 2{\cdot}0$)?

(*c*) What is the value of the force when the condenser is first filled up with petroleum and then charged?

41. (*a*) What is the value of the force which acts on the plates of the plate condenser in the preceding problem, if the condenser is separated from the battery after charging, and a plate of paraffin ($K = 2{\cdot}0$) 1 mm. in thickness is then inserted so as just not to touch the plates?

(*b*) What is the value of the force, if the paraffin plate is inserted before charging the condenser?

42. In the Gaussian system of units the sources of the electric field are given by the relation $\operatorname{div} \mathbf{E} = 4\pi\rho$. The units of the Heaviside-Lorentz "rational" system are defined in such a way that the factor 4π drops out of this equation, so that $\operatorname{div} \mathbf{E} = \rho$.

What is the constant factor for Coulomb's law in the rational system? What is the relation between the rational and Gaussian units of charge, intensity, pressure, and capacity?

43. Two condensers, whose capacities are $C_1 = 0{\cdot}5$ and $C_2 = 0{\cdot}2$ microfarads, are connected in series to a source of continuous pressure of 220 volts. What is the charge on the coatings, and what are the pressures in the two condensers?

44. Two equal air condensers, originally uncharged, are connected in series to a battery of pressure V. Find the change of potential in the wire connecting the two condensers, when one of the condensers is filled with a liquid of dielectric constant K.

45. Two condensers of capacities $C_1 = 1$ and $C_2 = 10$ microfarads, originally uncharged, are set up in series and connected to a battery the poles of which have potentials $+100$ and -100 volts relative to the earth. If the wire connecting the condensers is then earthed, find the quantity of electricity which is discharged through the earthing wire.

46. Assuming that the potential problem has been solved for the case of two infinitely long cylindrical conductors, one within the other, i.e. that the potential $\varphi(x, y)$ has been found, which takes constant values at the surfaces of the cylinders, and which satisfies Laplace's equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

at all points of the space between them—how is the mutual capacity of the two cylinders obtained?

47. In a plane problem, let the cross-sections of two cylindrical conductors be given, and a family of equipotential lines drawn, so that the difference of potential for any two adjacent lines is constant. How can we calculate from this diagram the mutual capacity of the two conductors (subject to the error arising from the limited number of the equipotential lines)?

48. By means of the function $w = u + iv = f(z)$ of the complex variable $z = x + iy$, a value $u = u(x, y)$ and a value $v = v(x, y)$ are associated with every point of the plane of xy. If the limit

$$\operatorname*{Lim}_{z_1 \to z} \frac{f(z_1) - f(z)}{z_1 - z} = \frac{dw}{dz}$$

exists, so that in particular it is independent of the path along which z_1 approaches z, prove that both u and v satisfy Laplace's equation, i.e. that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

and similarly for v.

In other words, if we have two curves, 1 and 2, on which e.g. u takes the constant values u_1 and u_2, then u is the solution of the potential problem for

a plane (cylindrical) field, which is generated by the two cylindrical conductors with the cross-sections 1 and 2, when the pressure between these is $u_1 - u_2$. If we multiply u by a suitable constant, we obtain the distribution of potential for a prescribed difference of potential between 1 and 2.

49. The complex function

$$z = x + iy = F(w) = F(u + iv)$$

(for which we assume the existence of the derivative $dz/dw \neq 0$, cf. Ex. 48) associates a point (x, y) of the z-plane with every point (u, v) of the w-plane. In the z-plane let a system B of conductors (or rather, of cross-sections of cylindrical conductors) be given, for which the potential problem has been solved, i.e. for which a function ϕ of x and y has been found which satisfies Laplace's equation and takes constant values at the given curves B. Let ϕ be the real part of the function $f(z) = \phi + i\psi$. Further, in the w-plane let a system A be given, for which the solution of the potential problem is wanted. Prove the following: *if a function $F(w)$ is known which represents the w-plane on the z-plane in such a way that it makes the figure A correspond to the figure B, then the real part of $f\{F(w)\} = f\{F(u + iv)\}$ is the potential function required for the system A.*

What is the ratio of the capacity of two conductors in A to that of the conductors corresponding to them in B?

50. Use the method of conformal representation (Ex. 49) as follows to determine the capacity (in cm.) of a twin circuit, i.e. of two parallel circular cylinders of the same radius r, the mean distance d between which is very great compared to r. Begin with a condenser formed by two concentric circular cylinders; then, by means of the function $w = 1/z$, with proper choice of the origin, represent this conformally on the given twin circuit. (In a transformation defined by $w = 1/z$, all circles remain circles, except such as pass through the origin, which become straight lines. The capacity of a cylindrical condenser is $1 \div (2 \log b/a)$, where a and b are the radii of the cylinders.)

51. From the fact that $1/r$ is a solution of the potential equation $\Delta\phi = 0$, deduce that

$$f = a\frac{\partial}{\partial x}\left(\frac{1}{r}\right), \quad g = b\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right), \quad h = c\,\frac{\partial^2}{\partial x \partial z}\left(\frac{1}{r}\right)$$

are also solutions of $\Delta\phi = 0$; a, b, c being constants.

By what arrangements of point charges near the origin are the potentials f, g, h respectively produced, at a great distance? [Two point charges for f, four for g or h.]

Electrodynamics

52. What is the length of tungsten wire in an incandescent lamp, if the lamp consumes 50 watts at 220 volts, and the diameter of the wire is 25 μ? [The specific resistance of the tungsten may be taken as proportional to the absolute temperature; at 18° C. it is $0{\cdot}056 \times 10^{-4}$ ohm-cm. The temperature of the wire may be put at 2500° abs.]

53. A tungsten filament lamp takes 50 watts at 220 volts. Find the value of the current immediately after the lamp has been switched in. How many times greater is it than the working current? The room temperature is 18° C. [For other data see Ex. 52.]

54. In an electric circuit the copper wire (cross-section 1 sq. mm.) is protected by a fuse of silver wire, the diameter of which is 0·2 mm. Calculate approximately (neglecting heat lost by conduction, &c.) how long it will take

for the fuse to be completely melted by a short-circuit current of 20 amperes; also the rise of temperature in the copper wire. [The specific heat of silver is 0·055 cal./gr. °C.; its specific resistance is $0·016 \times 10^{-4}$ ohm-cm.; melting-point 961° C. Copper has the specific heat 0·091 cal./gr. °C, and the specific resistance $0·017 \times 10^{-4}$ ohm-cm.]

55. A twin conductor (copper cross-section $= 2 \times 1$ mm.²) 300 m. in length connects a 220-volt generator to a point where the current is taken by two 100-watt lamps. By how much will the pressure at the lamps be lowered, if an electric box-iron consuming 500 watts is inserted in the circuit?

56. The following consuming devices are connected to a 220-volt supply: 6 lamps for 220 volts, 50 watts; 1 lamp for 8 volts and 6 amperes, with the requisite inserted resistance; and a motor, which at 220 volts develops $\frac{1}{6}$ h.p., with an efficiency of 75 per cent. Find the total resistance of the load, the current taken, and the power used. What resistance is needed for the 8-volt lamp?

57. An electric kettle taking 3 amperes at 220 volts brings 1 litre of water from 18° C. to boiling-point in 11 minutes. Find its efficiency, i.e. the percentage of the energy supplied which goes towards heating the water.

58. (*a*) The internal resistance of an ammeter is R. What resistance must be inserted in parallel to multiply the working range of the instrument n times?

(*b*) The internal resistance of a volt-meter is R. What resistance must be inserted in series to multiply the working range of the instrument n times?

59. The internal resistance of an accumulator is R, and the E.M.F. is V; and n of these are available. If the battery is divided into groups, each consisting of k accumulators connected in series, and the (n/k) groups thus arising are arranged in parallel, find the value of k for which the current is a maximum, the external resistance being R_1.

60. The deflection of a suspended coil galvanometer is proportional to the current and to n, the number of turns in the coil. Since the space available for the winding is fixed, we have a choice between a few turns of thick wire and a large number of turns of thin wire; the product of the number of turns n and the cross-section of the wire A is approximately constant, say $nA = F$. How should the coil be wound in order that, for a given E.M.F. (V), and a given external resistance (R), the deflection of the galvanometer may be a maximum?

61. The capacity C of an arrangement of two metallic conductors of any given shapes (or the capacity of a conductor standing by itself) is known. Suppose now that a medium of specific resistance ρ ohm-cm. is introduced into the space between the conductors, so as to fill it completely; ρ is much greater than the specific resistance of the metals, so that the pressure drop in the metal electrodes can be neglected. Determine the resistance R of the arrangement for the passage of current from the one conductor to the other.

62. An apparatus is earthed by means of a hemispherical metal electrode, which is sunk in the earth in such a way that the circular base lies in the earth's surface. The specific resistance of the ground is 10,000 ohm-cm. Find the earthing resistance. (Radius of electrode = 10 cm.)

63. A Daniell cell consists of two concentric cylinders, one of copper (radius a), the other of zinc (radius b); its height is h. If the specific resistance of the acid solution of copper sulphate is ρ ohm-cm., what is the "internal resistance" of the cell? (The capacity of a cylindrical condenser is $1 \div (2 \log b/a)$, where a and b are the radii of the cylinders.)

64. The interior of a plate condenser (plate distance d) is composed of two layers having respectively the thicknesses d_1 and d_2, dielectric constants K_1 and K_2, and conductivities σ_1 and σ_2; and $d = d_1 + d_2$. The pressure between the plates is V. Find the intensities $\mathbf{E}_1$ and $\mathbf{E}_2$, and the displacements $\mathbf{D}_1$ and $\mathbf{D}_2$. Find also the densities of the true and the free charge on the plane surface bounding the two layers. Determine the strength of the current through the condenser, and discuss the limiting case $\sigma_1 = 0$.

65. A plate condenser is filled with material of conductivity σ and dielectric constant K. By momentary contact with the terminal of a battery it is charged to potential V. Calculate the "time of relaxation", i.e. the time after which the charge (or pressure) of the condenser has fallen to the fraction $(1/e)$ of its original value.

66. The equation $\rho = \rho_0 e^{-4\pi\sigma t/K}$ is always true (as the proof on p. 115 shows), provided σ and K are constant in the space considered. Suppose e.g. that at time $t = 0$ a definite quantity of electricity was concentrated within a very small sphere, while the remainder of the medium carried no charge; then the equation states that every part of the medium which was originally uncharged, even if it is in the immediate neighbourhood of the charged part, remains electrically neutral while the charge on the sphere is disappearing. A quantity of heat existing in the small sphere at $t = 0$ would behave quite differently; it would *flow out* into the surrounding part of the medium, instead of merely fading away where it stands, like the electric charge. Explain this difference between the two cases.

67. A very large sphere of conductivity σ and dielectric constant K is placed in a vacuum; at its centre, as in the preceding problem, at $t = 0$, a given charge is confined within a very small sphere. Since the total charge of the system must remain constant, the exponentially vanishing charge at the centre must begin even from the very first to appear at the surface of the large sphere, no matter how great the radius of the latter may be. Prove that this phenomenon cannot be employed to transmit signals with infinite velocity, so that in spite of appearances the relativity postulate is not contradicted.

68. A large sphere of radius b consists of material of conductivity σ and dielectric constant K. At time $t = 0$ a charge Q is uniformly distributed over the surface of a small concentric sphere of radius a. Calculate the Joule heat developed during the discharge, and prove that it is equal to the loss of electrostatic energy due to the dispersal of the charge.

69. Find the intensity of the field generated by a dipole $\mathbf{m}$, at the end of a radius vector $\mathbf{r}$ drawn from the dipole.

70. The magnetic field of the earth can be represented with good approximation as the field of a magnetic dipole. Find the moment of this dipole, having given that the mean value of the horizontal intensity in magnetic latitude 45° is $H = 0{\cdot}23$ gauss. What relation must exist between the horizontal intensity at the magnetic equator and the vertical intensity at the magnetic poles?

71. Making the same assumption as in question 70, find how the dip i depends on the magnetic latitude.

72. The molecules of a paramagnetic gas are small magnetic dipoles of moment μ. In a homogeneous magnetic field $\mathbf{H}$ they would set themselves in the direction of the field, were they not disturbed by the thermal motion. As the effect of the thermal motion, at a given moment molecules of every orientation are present, but there is a smaller number of dipoles in directions differing greatly from the direction of the field. The distribution of the dipoles round the direction of the field is given by the Maxwell-Boltzmann formula

$$dn = Ae^{-E/kT}\, d\omega;$$

where dn is the number of dipoles with directions lying within the solid angle $d\omega$; E is the energy of a dipole making the angle α with the field ($E = -\mu H \cos \alpha$); k is the Boltzmann constant $= 1{\cdot}36 \times 10^{-16}$ ergs/°C.; and the constant factor A is defined by the condition that the integral of dn over all directions in space (i.e. over the whole solid angle 4π) must be equal to the total number of the molecules.

Find the value of the ratio defining the degree of magnetic saturation, viz.

$\sigma : \sigma_0 =$ magnetic moment of 1 gr. of gas

: magnetic moment of 1 gr. of gas, when the dipole axes are all in the direction of the field, the temperature being T, and the intensity H.

73. The result in the preceding question is given by "Langevin's formula"

$$\frac{\sigma}{\sigma_0} = \coth a - \frac{1}{a},$$

where $$a = \frac{\mu H}{kT}.$$

Find the limiting value of σ/σ_0 for $a \ll 1$, and for $a \gg 1$, and show that Curie's result $\varkappa = C/T$ (p. 133) is a good approximation to Langevin's when a is small. How are μ and σ_0 connected with Curie's constant C? Calculate the magnetic moment of a gramme molecule in terms of Curie's constant.

74. An iron sphere of radius $a = 5$ cm. is homogeneously magnetized to saturation (an ideal hard magnet; for iron $4\pi I_\infty = 22{,}000$). Find its dipole moment. What are the values of $\mathbf{B}$ and $\mathbf{H}$ in the sphere? Find the distribution of the surface divergence of $\mathbf{I}$ and the density of the "free" current on the surface of the sphere. What is the maximum value of the free current in amperes per cm.?

75. Calculate the field in external space due to the magnetized iron sphere of question 74.

76. The radius vector from one dipole $\mathbf{m}_1$ to another $\mathbf{m}_2$ is $\mathbf{r}$. Find the mutual energy of the two dipoles.

77. A magnetic needle of moment $|\,\mathbf{m}\,| = 100$ magnetic units floats horizontally on a cork in the sea. What are the positions of stable and of unstable equilibrium? What is the difference of energy in these two positions?

Give the answers also for the case when the needle is stuck in the cork vertically.

Take the earth's magnetic field (as in question 70) as due to a dipole; the magnetic moment of the earth is $8{\cdot}33 \times 10^{25}$.

78. The radius vector from the dipole $\mathbf{m}_1$ to the dipole $\mathbf{m}_2$ is $\mathbf{r}$. What is the force on $\mathbf{m}_2$?

How does the force between two small, freely movable magnetic needles depend on their distance apart?

79. A coil of 600 turns is wound round an iron ring of 20 cm. diameter and 10 sq. cm. cross-section; find the magnetic flux $\int B_n\, dA$ in the ring, when the current in the coil is 1 ampere. (Take $\mu = 500$.)

80. Suppose the iron ring of question 79 to contain an air gap of width δ, where δ is so small that it is not necessary for our purpose to take into account the spreading of the lines of induction in the air gap. How does the magnetic flux depend upon δ? What is its value for $\delta = 0{\cdot}1$, 1 and 5 mm.?

81. Calculate the field energy in the iron, the field energy in the air gap, the total field energy and the self-inductance (in henrys) of the divided iron ring of question 80, for the three widths of gap stated there.

82. Find the force with which the poles of the divided iron ring of question 80 attract each other. Discuss the energy balance of the process, supposing the poles to be drawn apart slightly. (Neglect elastic energy in the iron.)

83. Find the force (per cm.) with which the two wires of a twin circuit repel each other, when they are 30 cm. apart, and are carrying a current of 50 amps.

84. A suspended coil galvanometer has a square coil of 2 cm. side, and 100 turns; the coil can rotate about a vertical axis; the restoring couple due to the suspension is 10^{-2} gm.-cm./degree. The vertical sides of the coil are placed in a field of 1000 gauss, which is directed radially relative to the axis of rotation. Find the angle of deflection per milliampere. If the instrument is used as a mirror galvanometer, what is the current corresponding to a deflection of 1 mm. on a scale at a distance of 2 m.?

85. A string galvanometer consists of a thin stretched wire, set vertically and carrying a current in a homogeneous horizontal magnetic field. The deflection at the middle of the wire perpendicular to the lines of force is observed with a microscope. Find its value when the current is 1·0 milliampere, the length of the wire 5 cm., the (elastic) tension of the wire 0·02 gm., and the magnetic intensity 200 gauss. The form of the wire when the current is passing is a parabola with the equation (referred to its vertex as origin) $y = x^2(p/2Z)$, where p is the transverse force per unit length, and Z is the longitudinal tension.

86. A straight wire of length and direction **s** moves with velocity **u** in the magnetic field **B**. Its ends are connected by means of movable contacts with a fixed conductor, which with the wire forms a closed circuit. Find (in volts) the electromotive force induced in this circuit.

87. The two rails of a railway track are insulated from one another and from the ground (say by sleepers impregnated with oil), and connected through a millivoltmeter. What is the reading of the instrument when a train is passing at 100 km. per hour? The vertical component of the earth's magnetic field is 0·15 gauss, and the distance between the rails is 1435 mm.

88. A ring of copper wire of 20 cm. diameter and 1 mm.^{2} cross-section (sp. res. = $1{\cdot}75 \times 10^{-6}$ ohm-cm.) rotates in the earth's field about a vertical axis at 300 revolutions per minute. Find the Joule heat developed per second. Find also the average and maximum values of the necessary applied torque. (The horizontal component of the earth's magnetic field is 0·18 gauss.)

89. How does the current strength in the copper wire ring of question 88 depend on the angle ωt between the earth's field and the normal to the plane of the ring? Find the intensity (as a function of ωt) at the centre of the ring due to the current.

90. In a suspended coil galvanometer there are given: (1) the resistance R, breadth b, height l, and number of turns n, of the coil; (2) the (radial) magnetic field strength H in the air gap; (3) the (half) period of oscillation on open circuit; (4) the deflection C in degrees, for a current of 1 ampere. Calculate from these data the external resistance requisite for the non-periodic limiting condition.

91. An air choking coil of 0·3 henry self-inductance and 20 ohms virtual resistance is connected to an alternating pressure of 220 virtual volts at 50 cycles per second. Find the quantity of heat (in cal.) developed in the coil per minute.

92. A resistance of 10 ohms, a coil of self-inductance 0·5 henry, and a condenser of capacity 0·5 microfarad are joined up in series, and connected to a sinusoidal alternating pressure of 220 virtual volts and frequency 50 cycles per second. Find the current in virtual amperes, its phase displacement relative to the pressure, and the power.

93. The pressure of an alternator is not purely sinusoidal, but besides the fundamental of frequency ν contains also its third and seventh harmonics (frequencies 3ν and 7ν). The amplitude of the third harmonic is 5 per cent, that of the seventh 1 per cent of the amplitude of the fundamental. Find the amplitudes of the two harmonics (expressed as percentages of the fundamental amplitude) in the current, for the two cases when the generator is connected

(*a*) through a choking coil of negligible ohmic resistance;

(*b*) through a condenser.

94. (*a*) Three conductors are connected with each other at a "star-point". In them flow sinusoidal alternating currents of the same frequency and equal amplitudes; but the phase in each conductor is displaced by $2\pi/3$ relative to the preceding conductor (star connexion in three-phase system). Prove that the sum of the currents reaching the star-point at any moment is zero.

(*b*) Three coils are set up symmetrically in star formation about a central point in which their axes intersect, and the currents of a three-phase system flow through them. Each coil produces at the centre of the star a magnetic intensity of amplitude H which changes sinusoidally with the time, and has its direction along the axis of the coil; so that the fields due to the three coils are displaced by $2\pi/3$ relative to each other, in phase with respect to time, and in direction with respect to space.

Determine how the resultant intensity at the centre of the star varies with the time.

95. A choking coil of self-inductance $L = 1$ henry and resistance $R = 1$ ohm is connected, at time $t = 0$, to a battery of constant electromotive force V. Find the current at any time t. How long does it take before the current acquires its stationary value within 1 per cent?

96. A resonator (in the form frequently used to demonstrate electrical oscillations in the earlier experiments) consists of a circular ring (radius $R = 5$ cm.) of copper wire (diameter $2r = 1$ mm.); the ring is not quite complete and has two parallel circular metal plates (diameter $a = 5$ cm.) attached to its ends; these represent the capacity of the circuit, like the coatings of a plate condenser. When the resonator is placed in an alternating field, whose frequency agrees approximately with the proper frequency of the resonator, sparking occurs across the spark gap between the plates.

Find what the distance between the plates must be to demonstrate electrical waves of wave-length $\lambda = 1$ m.

97. (*a*) The energy, in the form of solar radiation, which is incident on an area of 1 sq. cm. perpendicular to the rays at the earth's surface, has the value 2·2 cal. (the "solar constant"). Calculate the root-mean-square values of the electric and magnetic intensities in sunlight, in volts per cm. and gauss respectively.

(*b*) Find the root-mean-square values of the electric and magnetic intensities in the radiation from a 50-watt lamp at a distance of 1 m., assuming that the lamp emits all the energy supplied to it.

98. A ring carrying a current produces at a great distance the same field as a "magnetic dipole". Assuming this equivalence, and using Maxwell's equations, deduce the field produced at a great distance by an alternating current in a coil, from the formulæ for an oscillating electric dipole. (Radiation from a "frame aerial".)

99. The plane wave

$$E_y = a \sin 2\pi\nu(t - x/c)$$
$$H_z = a \sin 2\pi\nu(t - x/c)$$

falls on the plane surface $x = 0$ of a conducting body extending indefinitely towards the side for which x is positive.

Find the pressure of radiation (p) due to the wave. Deduce the pressure (in kg./cm.^{2}) exerted by the sun's radiation on the earth's surface (at perpendicular incidence, and neglecting reflection). (For the value of the solar constant, see question 97.)

100. In the interior of an absorbing body the $\mathbf{E}$-vector of a light wave produces a current density $\mathbf{i}$, which combined with the $\mathbf{H}$-vector gives rise to a force density $\mathbf{f} = [\mathbf{iH}]/c$.

Prove from Maxwell's equations that the radiation pressure (p) in question 99 is identical with the volume integral of $\mathbf{f}$. $(p = \int_0^\infty f_x dx = a^2/8\pi.)$

Additional Miscellaneous Examples

101. Find a vector $\mathbf{v}$ such that $\operatorname{div} \mathbf{v} = 0$ everywhere, and $\operatorname{curl} \mathbf{v} = 0$ outside the cylinder $x^2 + y^2 = a^2$, while inside the cylinder $\operatorname{curl} \mathbf{v}$ is of constant magnitude ω and is in the direction of z.

102. The radii of three concentric thin spherical conducting shells are a, b, c (where $a < b < c$), and they carry charges Q_1, Q_2, Q_3 respectively. Write down without proof the potentials of the three shells.

If the outer shell is now connected to earth, find the changes in the potentials, and prove that the loss of energy is $\frac{1}{2}(Q_1 + Q_2 + Q_3)^2/c$.

103. Find an expression for the capacity per unit length of a long cylindrical condenser. How must the dielectric constant of the medium between the cylinders vary with the radius in order to ensure constant electric intensity between them?

104. A condenser is formed of three concentric cylinders of which the inner and outer are connected together. Obtain a formula for the capacity, neglecting end effects, and show that if the middle plate is 10 cm. long, and the radii of the cylinders are 3·9, 4·0, 4·1 cm., the capacity is approximately equal to that of a sphere of 4 m. radius.

105. A long thin insulated wire, having a charge of 6 e.s.u. per metre of its length, is stretched parallel to, and 2 m. distant from, an earthed conducting plane. Find the force per metre of its length with which the wire is attracted to the plane.

106. A conducting hemisphere of radius $3a$ is placed with its base in contact with an infinite conducting plane which is maintained at zero potential. If a point charge is placed on the axis of the hemisphere at a distance $4a$ from its centre and on the same side of the plane as the hemisphere, show that the total induced charge on the plane is to the total induced charge on the hemisphere in the ratio of 7 : 13.

107. Prove that the functions xy and $xy/(x^2 + y^2 + z^2)^{5/2}$ both satisfy Laplace's equation $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = 0$. Electricity is distributed on the surface of the sphere $x^2 + y^2 + z^2 = a^2$ so that the surface density at the point (x, y, z) is xy. Using the above result, find the potential at any point inside or outside the sphere.

108. If λ is a given function of x, y, z, prove that a solution of Laplace's equation exists in the form of a function of λ, provided the ratio

$$\left(\frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2} + \frac{\partial^2\lambda}{\partial z^2}\right) : \left\{\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2 + \left(\frac{\partial\lambda}{\partial z}\right)^2\right\}$$

is a function of λ. Illustrate by the case $\lambda = x^2 + y^2 + z^2$.

109. If λ is the positive root of the equation $x^2 + y^2 = 4\lambda(z + \lambda)$, prove that λ satisfies the condition of Ex. 108. Hence show that confocal paraboloids of revolution form two families of equipotential surfaces.

110. An infinite hollow circular cylinder of soft iron, of permeability μ, whose internal and external radii are a and b, is placed in a uniform magnetic field of strength H perpendicular to the axis of the cylinder. Show that the magnetic force within the cavity is $4\mu b^2 H/\{(\mu^2 + 1)(b^2 - a^2) + 2\mu(b^2 + a^2)\}$.

111. A small spherical cavity is made in a magnet. Prove that the magnetic force at the centre of the cavity is compounded of $\frac{2}{3}\mathbf{H}$ and $\frac{1}{3}\mathbf{B}$, where $\mathbf{H}$ and $\mathbf{B}$ are what the magnetic force and induction at the same point would be if there were no cavity. (The magnet is ideally hard.)

112. A very small magnet of moment M is placed at a height d above the horizontal plane face of a block of iron, which may be supposed to extend to infinity laterally and downwards. Show that the magnet is attracted to the block with a force $\frac{3}{8}\,\dfrac{\mu - 1}{\mu + 1}\,\dfrac{M^2}{d^4}$, where μ is the permeability of the iron, and the axis of the magnet is vertical.

113. A sphere of soft iron of radius a and permeability μ is placed in the field due to a single magnetic pole of strength m at a distance f from the centre of the sphere. Prove that if a is small compared with f, the sphere will be attracted to the pole with a force approximately equal to $2\,\dfrac{\mu - 1}{\mu + 2}\,\dfrac{m^2 a^3}{f^5}$.

114. Find the relative amounts of energy stored in a 2-volt accumulator which gives 100 ampere-hours discharge, and in a condenser of a million e.s.u. of capacity, charged to 100,000 volts.

115. A copper cylinder of length 50 cm. and internal diameter 6 cm., having a glass bottom, is filled with copper sulphate solution, and has a copper wire of length 50 cm. and diameter 1 mm. placed down its axis. The specific resistance of the solution is 33 ohms per cm. cube. Find the current passing through the solution if 2 volts potential difference is maintained between the cylinder and the wire.

116. An infinite straight wire is coplanar with a wire in the form of a circle of radius a, the wires not meeting. Prove that the coefficient of mutual induction of the wires (in e.m.u.) is $4\pi\{h - \sqrt{(h^2 - a^2)}\}$, where h is the distance of the straight wire from the centre of the circle.

117. Prove that the vector potential due to a circular current, at a point distant ρ from the axis of the circle and z from its plane, is directed at right angles to the axial plane through the point, and is of magnitude $ia\int_0^{2\pi}\cos\theta\, d\theta/D$, where i is the current in e.m.u., a the radius of the circle, and $D^2 = a^2 + \rho^2 + z^2 - 2a\rho\cos\theta$.

118. A circular wire of radius a and resistance R is spun in a magnetic field of strength H with angular velocity ω about an axis which is in the plane of the wire and is perpendicular to the lines of force. Show that the average rate of dissipation of energy by electric currents induced in the wire is approximately $\frac{1}{2}\pi^2 a^4 \omega^2 H^2/R$. ($R$ is in e.m.u.)

119. A condenser of capacity 5 microfarads has imperfect insulation, the leak between the terminals being equivalent to 200 million ohms. If a constant charging current of 3 microamperes runs into the condenser, find how long it will take for the potential across the condenser to rise to 500 volts.

120. A circular coil of wire, of radius a and resistance R, is fixed in a plane, and a thin bar magnet, of moment M and length $2b$, is perpendicular to this plane and moves along the axis of the coil. Show that, when the centre of the magnet is at a small distance x from the centre of the coil, the number of lines of magnetic induction passing through the coil is approximately (e.m.u. being used) $2\pi M(1/b - 1/f + 3a^2x^2/2f^5)$, where $f^2 = a^2 + b^2$.

Find the current induced in the coil if $x = x_0 \sin nt$.

121. The length, l cm., of a solenoid is great compared to its diameter. The area of the section is A sq. cm., and the number of turns n. Prove that the self-inductance L is $4\pi n^2 A/l$, in e.m.u.

What is the frequency of oscillation in a circuit of negligible resistance, consisting of a straight solenoid of 1000 turns, of length 50 cm. and radius 3 cm., in series with a capacity of 0·2 microfarad?

122. A conductor carrying a current i has the form of a plane curve, and two magnetic poles N, S of strengths m, $-m$ lie in its plane. Prove that the couple tending to turn any portion AB of the conductor about NS as an axis is $mi(\cos ANS + \cos ASN - \cos BNS - \cos BSN)$.

123. A circuit of self-inductance L and resistance R is connected at time x to mains of which the P.D. is given by $E \sin 2\pi nt$. Find a general expression for the value of the current at any subsequent time t. What must be the value of x in order that no transient current shall flow in the circuit?

124. Two coils of negligible resistance each contain a condenser of such capacity that the frequency of the electromagnetic oscillations in each coil is n, when the coils are apart; prove that, when the coils are close enough for the oscillations to be affected by mutual induction, the frequencies of the two principal oscillations become $n/(1 \pm k)^{\frac{1}{2}}$, where $k^2 = M^2/LN$; L, N, M being the self-inductances and mutual inductance.

125. Two precisely similar circuits each consisting of a condenser of capacity C short-circuited by a wire of self-inductance L and resistance R have each a period T when at a great distance from each other. Show that when they are brought near together so that the two circuits have a small mutual inductance M each circuit will have two periods given by $T\{1 \pm (M/L)(1 - T^2/8\pi^2 LC)\}$.

126. A circuit contains two impedances connected *in series*. Each impedance consists of a self-inductance, a resistance, and a capacity connected in series, the values being L_1, R_1, C_1 and L_2, R_2, C_2. Find the current due to a given sinusoidal pressure of frequency ω.

127. Find the current, for the circuit of Ex. 126, when the two impedances are connected *in parallel*.

128. An alternating current runs through a non-inductive resistance R. Find how much the voltage drop down R is reduced if a capacity C is placed in parallel with R, and the same current is made to flow through the combination. If $R = 30{,}000$ ohms, and $C = 300{,}000$ e.s.u., find for what frequency the reduction is 5 per cent.

129. Compare the effects of (i) an inductance and a capacity in series, (ii) the same in parallel, on the total current supplied by an alternating E.M.F. through each arrangement. Prove that when the circuits are "tuned" the total currents will tend towards infinity and zero respectively as the resistances are indefinitely diminished.

130. Two conductors of capacities C_1 and C_2 have one plate of each earthed. Initially the first condenser has a charge Q_0 and the second is uncharged. The two insulated plates are joined by a wire of resistance R and self-inductance L. Prove that if $R^2 = 4L(1/C_1 + 1/C_2)$ the current in the wire will be a maximum when $t = 2L/R$, the maximum value being $2Q_0/(eC_1R)$.

131. Discuss the values of the initial current, and of the total charge, sent round a circuit which is acted on by an electromotive impulse.

An electromotive impulse acts in a primary coil of self-inductance L. Show that by the presence in the neighbourhood of a closed secondary of self-inductance N, and having mutual inductance M with the primary, the effective initial self-inductance is reduced from L to $L - M^2/N$.

132. A helical coil (100 cm. long, 4 cm. diameter, 1000 turns) is overwound near its centre by a secondary coil (400 turns, 3 ohms resistance) which is short-circuited.

(1) What charge will flow round the secondary on making a steady current of 2 amps in the primary?

(2) What current will be induced in the secondary by a simple harmonic current of root-mean-square 2 amps and frequency 50 cycles in the primary?

133. Two points A and B are joined by two wires in parallel containing resistances R_1, R_2, and self-inductances L_1, L_2 respectively. Show that if a condenser is discharged by joining its poles to A and B, the total amount of electricity discharged by each wire is independent of L_1 and L_2.

134. A dynamo supplies an electromotive force $E \sin nt$ to a circuit of resistance R and self-inductance L. A circuit, also of resistance R and self-inductance L, is linked to the first, and the coefficient of mutual induction is L. Show that, when the currents in the two circuits are periodic, the current in the circuit containing the dynamo is

$$\frac{E}{R}\left(\frac{R^2 + L^2n^2}{R^2 + 4L^2n^2}\right)^{\frac{1}{2}} \sin\left(nt - \tan^{-1}\frac{LRn}{R^2 + 2L^2n^2}\right).$$

Show also that the mean rate at which the dynamo is supplying energy is $E^2(R^2 + 2L^2n^2)/2R(R^2 + 4L^2n^2)$.

135. If $V = \{f(t - r/c) + F(t + r/c)\}/r$, where f and F denote arbitrary functions, show that

$$E_x = \frac{\partial^2 V}{\partial x\, \partial z}, \quad E_y = \frac{\partial^2 V}{\partial y\, \partial z}, \quad E_z = -\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2}$$

are possible values of the electric force in free space. Show also that $V = r^{-1}e^{j\nu t}\sin(\nu r/c)$ gives a possible field inside a hollow perfectly conducting spherical shell of radius a and centre at the origin, if ν is a root of $\cot(a\nu/c) = c/a\nu - a\nu/c$.

136. A current flows in a straight wire of circular section, and the current density (i in e.m.u.) at time t at a distance r from the axis is of the form $f(r, t)$. Neglecting the displacement current, prove that $4\pi ri = \dfrac{\partial}{\partial r}(Hr)$, $\dfrac{\partial E}{\partial r} = \mu\dfrac{\partial H}{\partial t}$, where E is the electric and H the magnetic vector.

137. In Ex. 136, assume that the variables involve the time only in the form of a factor $e^{j\omega t}$. The wire is of radius a, conductivity σ and permeability μ. Show that $\dfrac{\partial}{\partial r}\left(r\dfrac{\partial i}{\partial r}\right) = (4\pi j\mu\sigma\omega)ri$.

If $\pi\mu\sigma\omega a^2$ is a small fraction, prove that an approximate solution is $i = Ae^{j\omega t}(1 + \pi j\mu\sigma\omega r^2)$, where A is a constant. If I_0 is the effective total current, show that the heat developed in the wire per unit time and unit length is $(I_0{}^2/\pi\sigma a^2)(1 + \frac{1}{12}\pi^2\mu^2\sigma^2\omega^2 a^4)$.

ANSWERS TO EXAMPLES
WITH HINTS FOR SOLUTION

1. Components of $\mathbf{A}+\mathbf{B}$: $-1{\cdot}23, +6{\cdot}97, +8{\cdot}33$; $|\mathbf{A}+\mathbf{B}| = 10{\cdot}9$; $\alpha = 96{\cdot}5°$, $\beta = 50{\cdot}4°$, $\gamma = 40{\cdot}3°$.

2. Work done by the force $= -135$ kg.-cm., i.e. 135 kg.-cm. of work are done against the force.

3. $|\mathbf{ab}| = 1$; $\alpha = 70{\cdot}5°$.

4. Scalar product $= 1$, angle $= 60°$; the six possible vector products are $\pm(-\mathbf{i}, \mathbf{j}, \mathbf{k})$, $\pm(\mathbf{i}, -\mathbf{j}, \mathbf{k})$, $\pm(\mathbf{i}, \mathbf{j}, -\mathbf{k})$; area of face of tetrahedron $= \frac{1}{2}$ absolute value of vector product $= \sqrt{3}/2$.

5. $-\sqrt{3}, +\sqrt{3}, -\sqrt{3}$.

6. The equation represents a plane, since it is linear in the co-ordinates of P; the plane passes through the end of $\mathbf{a}$, since it is satisfied by $\mathbf{r} = \mathbf{a}$; it is perpendicular to $\mathbf{a}$, since all other vectors which satisfy the equation, but have a different direction from $\mathbf{a}$, must be longer, in order to give the scalar product the value assigned.

7. $9{\cdot}3, 9{\cdot}3, 0$.

8. Follows immediately from the formula for $[\mathbf{A}[\mathbf{BC}]]$.

9. $-\frac{1}{3}, -\frac{1}{3}, +\frac{2}{3}$.

10. The components of the unit vectors are, in order: $(0{\cdot}816, 0{\cdot}408, 0{\cdot}408)$; $(-0{\cdot}566, 0{\cdot}707, 0{\cdot}424)$; $(0{\cdot}115, 0{\cdot}577, -0{\cdot}808)$.

11. $(11{\cdot}4, 7{\cdot}1, -2{\cdot}9)$.

12. The distance d is $D/\sqrt{(k^2 + l^2 + m^2)}$, where

$$D = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix},$$

$$\begin{aligned} k &= (b_y c_z - b_z c_y) + (c_y a_z - c_z a_y) + (a_y b_z - a_z b_y), \\ l &= (b_z c_x - b_x c_z) + (c_z a_x - c_x a_z) + (a_z b_x - a_x b_z), \\ m &= (b_x c_y - b_y c_x) + (c_x a_y - c_y a_x) + (a_x b_y - a_y b_x). \end{aligned}$$

Geometrically interpreted, the formula gives the height of a tetrahedron in terms of its volume and base-area. (Cf. questions 15 and 16.)

13. 6·3. 14. (−43·1, −35·0, +58·1) cm./sec. 15. 21. 16. 13.

17. Of all parallelepipeds having edges of given lengths, the one which is rectangular has the greatest volume. Of all triangles with two given sides, that one has the greatest area in which these two sides are at right angles.

18. Since $\mathbf{t}^2 = 1$, it follows that $\left(\mathbf{t}\,\frac{d\mathbf{t}}{ds}\right) = 0$. $\frac{d\mathbf{t}}{ds}$ is the curvature, i.e. a vector to the centre of curvature, equal to the reciprocal of the radius of curvature.

19. $\operatorname{grad}(\mathbf{ar}) = \mathbf{a}$.

20. The direction of the vector is that of the line of intersection of the two surfaces $\psi =$ const., $\varphi =$ const. which pass through the point in question.

21. $\operatorname{curl}\mathbf{v} = 2\boldsymbol{\omega}$. 22. $\omega = k/r^2$. 23. $f(r) = \text{const.}/r^2$. 24. $f(r) = \text{const.}/r^3$.

25. $\mathbf{A} = [\mathbf{r}\,.\,\operatorname{grad} f]$, from which the results stated follow at once.

26. 917 kg. 27. $-\,e^2 r/a^3$.

28. $(\mathbf{mE}) = +\,|\,\mathbf{m}\,|\,.\,|\,\mathbf{E}\,|\,.\cos\alpha$. If the dipole can rotate freely, put $\cos\alpha = 1$.

29. (*a*) $6{\cdot}4 \times 10^{-5}$ gr.; (*b*) $4{\cdot}8 \times 10^{-5}$ gr.

30. Metal sphere, $\omega = (3E_0/4\pi)\cos\theta$; dielectric sphere, $\omega = \frac{3E_0}{4\pi}\,\frac{K-1}{K+2}\cos\theta$.

31. $\frac{e^2R^3}{d^3}\,\frac{2d^2-R^2}{(d^2-R^2)^2} - \frac{Ee}{d^2}$. 32. 1667 e.s.u. $= 5{\cdot}56 \times 10^{-7}$ coulomb.

33. $r = 5$ cm.

34. $\omega = 6{\cdot}6 \times 10^{-4}$ e.s.u./cm.^{2}; $F = 2{\cdot}8 \times 10^{-2}$ dyne/m.^{2}.

35. $r = 11$ mm. 36. $V^2 = 32\pi rT$. 37. $\mathbf{E} = \mathbf{E}_0$; $\mathbf{D} = K\mathbf{E}_0$.

38. (*a*) $\mathbf{E}$; (*b*) $K\mathbf{E} = \mathbf{E} + 4\pi\mathbf{P}$; (*c*) $\mathbf{E}\,\frac{3K}{1+2K} = \mathbf{E} - \frac{4\pi}{3}\mathbf{P}$.

39. In glass, $|\,\mathbf{E}\,| = \frac{V}{x+(d-x)K}$; in air space, $|\,\mathbf{E}\,| = \frac{KV}{x+(d-x)K}$.

40. (*a*) 44·3 dynes/cm.^{2}; (*b*) 22·1 dynes/cm.^{2}; (*c*) 88·6 dynes/cm.^{2}.

41. (*a*) 44·3 dynes/cm.^{2}; (*b*) 177·2 dynes/cm.^{2}.

42. Factor $= 1/4\pi$. The rational units of charge, field strength, pressure, and capacity are respectively $1/\sqrt{4\pi}$, $\sqrt{4\pi}$, $\sqrt{4\pi}$, and $1/4\pi$ times the corresponding Gaussian units.

43. $Q = 3{\cdot}15 \times 10^{-5}$ coulomb, 63 volts, 157 volts.

44. $\frac{1}{2}\,\frac{K-1}{K+1}\,V$. 45. 9×10^{-4} coulomb $= 2{\cdot}7 \times 10^6$ e.s.u.

46. $C = \frac{}{4\pi(\varphi_2 - \varphi_1)}\int \operatorname{grad}\varphi\,.\,ds$, per cm.

47. Since $4\pi\omega = |\partial\varphi/\partial n| \approx |\delta\varphi/\delta n|$, we have the following construction. Draw the lines of force (the orthogonal trajectories of the equipotential lines) in such a way that the distance between two consecutive lines of force at any point is equal to the distance between two consecutive equipotential lines at the same point (i.e. so that the network of the two sets of lines consists of *squares*). We take a length 1 cm. of the cylindrical system; hence in a tube of force defined by two lines of force the area of the section made by one of the conductors is equal to the breadth b of the tube of force at that conductor. The charge on this area is therefore $\frac{1}{4\pi}\left|\frac{\delta\varphi}{b}\right| b = \delta\varphi/4\pi$; and if m tubes of force issue from the conductor, the total charge per unit length is $m\,\delta\varphi/4\pi$. If further the pressure V between the conductors is divided up into n parts by the equipotential lines, drawn as explained, we have $\delta\varphi = V/n$, so that the capacity C is $(1/4\pi)(m/n)$ electrostatic units.

48. Form the limit, (a) with real, (b) with pure imaginary $z_1 - z$, and equate the two results. We thus obtain the Cauchy-Riemann differential equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. By differentiating these with respect to x, y respectively, and adding, we obtain Laplace's equation for u; and similarly for v.

49. For the first part of the question it is sufficient to remark that, subject to the conditions stated, $f\{F(w)\}$ is also a function of w with the requisite properties of regularity, and that its real part also is constant on the corresponding curves of A. The capacities of the original and transformed systems are equal, as may be proved as follows. Any complex number, and therefore in particular the derivative of $F(w)$ at the point P of the w-plane, can be written in the form $re^{i\phi}$, so that $dz = re^{i\phi}\,dw$; this states that the transform dz of any infinitely small vector dw is derived from dw by a stretch in the ratio $r : 1$ and a rotation through the angle $d\varphi$. Hence the angle between two linear elements passing through P, and the ratio of their lengths, both remain unchanged; on this account this representation by means of a differentiable complex function is referred to as "conformal". The network of squares formed by the equipotential lines and lines of force in B transforms therefore into a network of squares in A; and the number of lines in each set is of course not changed. These two facts, taken along with the answer to question 47, prove the result stated above.

50. To find an origin from which the circular sections of the two cylinders can be transformed into the concentric circles of the cylindrical condenser, construct any circle cutting the two circular sections at right angles. Either of the points ("limiting points") where this orthogonal circle cuts the line of centres of the sections will serve as origin. When $d \gg a$, the value of the capacity of the twin circuit becomes $1/\{4\log(d/a)\}$.

51. The potential f is produced by a dipole ($+e$ at $x = 0$ and $-e$ at $x = l$, with $el = a$); g by two oppositely directed dipoles on the x-axis, of moment a and at distance d, such that $ad = b$; h similarly by two dipoles on, but perpendicular to the x-axis.

52. 99 cm.

53. $i = 1{\cdot}95$ amperes; 8·6 times greater than the working current.

54. 0·35 sec.; 0·7° C. 55. 23·3 volts.

56. Current 8·1 amperes; load resistance 27·2 ohms; power 1780 watts; inserted resistance for 8-volt lamp 35·4 ohms.

57. 79 per cent. 58. (a) $R/(n-1)$; (b) $(n-1)R$. 59. $k = \sqrt{(nR_1/R)}$.

60. The resistance of the coil must be equal to the external resistance.

61. $R = \rho/4\pi C$. 62. $R = \rho/2\pi r$; if $r = 10$ cm., $R = 159$ ohms.

63. $R = \rho \log b/a \div 4\pi h$.

64. $E_1/V\sigma_2 = E_2/V\sigma_1 = D_1/K_1V\sigma_2 = D_2/K_2V\sigma_1 = i/V\sigma_1\sigma_2 = 1/(d_1\sigma_2 + d_2\sigma_1)$. Density of true charge $\pm(V/4\pi)(K_1\sigma_2 - K_2\sigma_1)/(d_1\sigma_2 + d_2\sigma_1)$; density of free charge $\pm(V/4\pi)(\sigma_2 - \sigma_1)/d_1\sigma_2 + d_2\sigma_1$. If the conductivity σ_1 of the first layer is zero, then there is no field in the other layer, and no charge on its plate; we have simply a condenser with plate distance d_1, dielectric constant K_1, and pressure V.

65. $K/4\pi\sigma$.

66. If e is the charge concentrated in the small sphere at time $t = 0$, this produces at distance r the field strength e/r^2, which calls forth the current density $i = \sigma e/r^2$; i is directed radially. The flow of electricity is therefore solenoidal; there is no condensation or rarefaction of conduction electrons, and consequently no charge originates anywhere. While, therefore, in conduction of heat or in diffusion the dispersing entity tends to spread outwards, here the action at a distance of the electric charge acts on the conduction electrons at different distances with forces which are exactly such as to have no concentrating effect. From a mathematical point of view, the force which produces the flow in heat conduction or diffusion is the gradient of the density of the flowing substance; in electricity, on the other hand, the application of the gradient operator to the driving force $\mathbf{E}$ gives the density ($\operatorname{div} \mathbf{E} = 4\pi\rho$).

67. In order to be able to give a signal at a definite moment, we must, until this moment arrives, prevent the charge which is concentrated on the small sphere from being dispersed, by means of an insulating envelope, which is withdrawn at the given moment. Before this, however, an induced charge equal and opposite to the charge on the sphere would appear on the external surface of the envelope, and at the same time an induced charge equal to the original charge would be produced on the surface of the large sphere. The withdrawal of the insulating envelope would merely cause the charges on its two sides to unite.

68. $\dfrac{Q^2}{2K}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$. 69. $\mathbf{H} = \dfrac{3(\mathbf{mr})}{r^5}\mathbf{r} - \dfrac{\mathbf{m}}{r^3}$.

70. $|\mathbf{m}| = 8{\cdot}37 \times 10^{25}$; the vertical intensity at the poles is double the horizontal intensity at the equator.

71. $\tan i = 2 \tan \beta$. **72.** See 73.

73. For $a \gg 1$, $\sigma/\sigma_0 \approx 1 - 1/a$; for $a \ll 1$, $\sigma/\sigma_0 \approx \frac{1}{3}a$. $C = \sigma_0 \mu \delta / 3k$ (δ = sp. grav. of the substance). The magnetic moment of a gramme-molecule is $\sqrt{(3MRC/\delta)}$, where M is the molecular weight and R the gas constant.

74. $|\mathbf{m}| = 916{,}000$; $|\mathbf{H}| = -7{,}350$; $|\mathbf{B}| = 14{,}700$. The surface divergence of $\mathbf{I}$ ("surface density of magnetism") is $-|\mathbf{I}|\cos\theta$; the surface density of the free current $i' = c|\mathbf{I}|\sin\theta$, where θ is the angle between the direction of magnetization and the spherical surface. The maximum value of the free current is 17,500 amperes/cm.

75. $H_r = (8\pi/3)a^3|\mathbf{I}|\cos\theta/r^3$; $H_\theta = (4\pi/3)a^3|\mathbf{I}|\sin\theta/r^3$.

76. $(\mathbf{m}_1\mathbf{m}_2)/r^3 - 3(\mathbf{m}_1\mathbf{r})(\mathbf{m}_2\mathbf{r})/r^5$.

77. The needle floating horizontally is in stable equilibrium at the magnetic equator, unstable at the magnetic poles; the energy difference at the two places is 32·3 ergs. The vertical needle is stable at that pole where its direction is the same as that of the earth-dipole, unstable at the other pole; the difference of energy is 129·2 ergs.

78. $\dfrac{3}{r^5}\left\{(\mathbf{m}_1\mathbf{r})\mathbf{m}_2 + (\mathbf{m}_2\mathbf{r})\mathbf{m}_1 + (\mathbf{m}_1\mathbf{m}_2)\mathbf{r} - \dfrac{5(\mathbf{m}_1\mathbf{r})(\mathbf{m}_2\mathbf{r})}{r^2}\mathbf{r}\right\}$. The force between two magnets which are free to rotate is $-6|\mathbf{m}_1|.|\mathbf{m}_2|/r^4$ (the negative sign indicating attraction).

79. 60,000 maxwells.

80. 55,500, 33,400, 12,100 maxwells, for $\delta = 0{\cdot}1$, $1{\cdot}0$, $5{\cdot}0$ mm.

81. Energy in iron 0·154, 0·055, 0·007 joules; energy in air gap 0·023, 0·044, 0·029 joules; total energy 0·177, 0·099, 0·036 joules; self-inductance 0·354, 0·198, 0·072 henrys—for $\delta = 0{\cdot}1$, 1, 5 mm.

82. Force 12·5, 4·5, 0·6 kg. for $\delta = 0{\cdot}1$, 1, 5 mm. If the poles are drawn a small distance apart, so that the self-inductance L of the divided ring is decreased by ΔL, the field energy, when the current i is kept constant, diminishes by $\frac{1}{2}i^2\Delta L$. The magnetic flux decreases by $\Delta\varphi = ci\,\Delta L$, causing an E.M.F. of self-inductance $V = -\dfrac{1}{c}\dfrac{d\varphi}{dt}$ in the same direction as the current, the total work of which, viz. $\int Vi\,dt = (i\,\Delta\varphi)/c = i^2\,\Delta L$, is restored to the battery which feeds the coil. The mechanical work expended in the process is therefore $\frac{1}{2}i^2\Delta L$, which is equal to the decrease of the field energy, so that in all double this amount, i.e. $i^2\Delta L$, is restored to the battery.

83. 1·67 dynes/cm.

84. $\alpha = 4{\cdot}1°$; 1 mm. on the scale corresponds to 0·0031 milliampere.

85. 0·032 mm. **86.** $\mathbf{s}[\mathbf{uB}]\,.\,10^{-8}$ volt. **87.** 0·6 millivolt.

88. $3{\cdot}4 \times 10^{-5}$ cal./sec.; average torque $4{\cdot}6 \times 10^{-6}$ gm.-cm.; maximum torque $9{\cdot}2 \times 10^{-6}$ gm.-cm.

89. If R is the resistance of the ring, $i = (\omega\pi r^2 H/R)10^{-8} \sin\omega t = 1{\cdot}62 \sin\omega t$ milliamperes; the intensity perpendicular to the plane of the ring is $H' = (2\pi^2\omega r H/R)10^{-9} \sin\omega t = 1{\cdot}02 \times 10^{-4} \sin\omega t$ gauss.

90. External resistance in ohms $= (\pi^2 C/360\tau) bln H \,.\, 10^{-8} - R$.

91. 1490 cal./min. **92.** 0·0354; 89° 54·5′ ahead; 0·0125 watt.

93. (*a*) 1·7 per cent; 0·14 per cent; (*b*) 15 per cent; 7 per cent.

94. (*b*) The magnitude of the intensity is constant; its direction rotates with angular velocity ω.

95. $i = i_0(1 - e^{-t/\theta})$; $i_0 = V/R$; $\theta = L/R$. Time = 6·9 sec.

96. $d = 3{\cdot}7$ mm.

97. (*a*) 7·5 volts/cm.; $2{\cdot}5 \times 10^{-2}$ gauss: (*b*) 0·39 volt/cm.; $1{\cdot}3 \times 10^{-3}$ gauss.

98. If $\mathbf{E}$ and $\mathbf{H}$ are solutions of Maxwell's equations for a vacuum, then these equations are also satisfied by the vectors $\mathbf{E}' = -\mathbf{H}$, and $\mathbf{H}' = +\mathbf{E}$. In equations (44*a*) and (44*b*), p. 225, replace the vector $\mathbf{E}$ by $\mathbf{H}$, and the vector $\mathbf{H}$ by $-\mathbf{E}$; we thus obtain the radiation from a magnetic dipole $\mathbf{p}$. For a flat coil of n turns and area S, carrying a current i, the value of p is nSi/c.

99. $a^2/8\pi$, i.e. for the sun's radiation, 5×10^{-11} kg./cm.^{2}.

100. From $\dfrac{\partial E_y}{\partial x} = -\dfrac{1}{c}\dfrac{\partial H_z}{\partial t}$, and $\dfrac{\partial H_z}{\partial x} = -\dfrac{4\pi}{c} i_y - \dfrac{1}{c}\dfrac{\partial E_y}{\partial t}$, it follows that

$$f_x = \frac{1}{c} i_y H_z = -\frac{1}{4\pi}\left(H_z \frac{\partial H_z}{\partial x} + \frac{1}{c} H_z \frac{\partial E_y}{\partial t}\right)$$
$$= -\frac{1}{4\pi}\left\{\frac{1}{2}\frac{\partial H_z^2}{\partial x} - \frac{1}{c} E_y \frac{\partial H_z}{\partial t} + \frac{1}{c}\frac{\partial}{\partial t}(H_z E_y)\right\},$$

so that, for the time average,

$$\overline{f_x} = -\frac{1}{8\pi}\frac{\partial}{\partial x}(H_z^2 + E_y^2), \text{ \&c.}$$

101. Inside, $v_x = -\frac{1}{2}\omega y$, $v_y = \frac{1}{2}\omega x$, $v_z = 0$; outside, $v_x = -\frac{1}{2}\omega a^2 y/(x^2 + y^2)$, $v_y = \frac{1}{2}\omega a^2 x/(x^2 + y^2)$, $v_z = 0$.

102. $V_1 = Q_1/a + Q_2/b + Q_3/c$, $V_2 = Q_1/b + Q_2/b + Q_3/c$, $V_3 = Q_1/c + Q_2/c + Q_3/c$. New potentials are $V_1 - V_3$, $V_2 - V_3$, 0. Loss of energy $= \frac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3) - \frac{1}{2}\{Q_1(V_1 - V_3) + Q_2(V_2 - V_3)\} = \frac{1}{2}(Q_1 + Q_2 + Q_3)V_3$.

103. (i) If $K = 1$, $E_r \,.\, 2\pi r =$ const., $\varphi = A \log(r/r_0)$, charge per unit length $= \frac{1}{2}A$, capacity per unit length $= 1 \div 2\log(r_1/r_0)$. (ii) If E_r is constant, then $Kr =$ const.

104. Let radii be a, b, c with $a < b < c$. Take $V = 0$ on outer and inner, $V = V_0$ on mid cylinder. Between a and b, $V = V_0 \log(r/a)/\log(b/a)$; between b and c, $V = V_0 \log(r/c)/\log(b/c)$. On inner side of b, $4\pi\omega_1 = \partial V/\partial r = V_0 \div \{b \log(b/a)\}$; on outer side of b, $4\pi\omega_2 = -\partial V/\partial r = V_0 \div \{b \log(c/b)\}$. Capacity $= 2\pi b l(\omega_1 + \omega_2)/V_0 = \frac{1}{2}l\{1/(\log b/a) + 1/(\log c/b)\}$. In numerical case, capacity is rather over 400 e.s.u.; capacity of sphere is 400 e.s.u.

105. Attraction is equal to that of image of charged line in the plane $= 2$(linear density on image) $\times$ (charge attracted) $\div$ (distance between the lines) $= 2 \times (6/100) \times 6 \div 400 = 18 \times 10^{-4}$ dyne.

106. The image system consists of three charges, $-e$ at height $-4a$, $-\frac{3}{4}e$ at $\frac{9}{4}a$, and $+\frac{3}{4}e$ at $-\frac{9}{4}a$. If ω is the induced density at any point of the plane, then $4\pi\omega =$ normal force due to the four charges. Now normal force due, say, to q at height h and $-q$ at $-h$, is $-2qh/(\rho^2 + h^2)^{3/2}$, and the corresponding induced charge on the plane is $(-1/4\pi)\int_{3a}^{\infty} 2\pi\rho \, d\rho \, 2qh/(\rho^2 + h^2)^{3/2}$ or $-qh/\sqrt{(9a^2 + h^2)}$. Thus total induced charge on plane is $-e \,.\, 4a/\sqrt{(9a^2 + 16a^2)} + \frac{3}{4}e \,.\, \frac{9}{4}a/\sqrt{(9a^2 + 81a^2/16)} = -\frac{4}{5}e + \frac{9}{20}e = -\frac{7}{20}e$; and, since the total induced charge is $-e$, the charge on hemisphere $= -e + \frac{7}{20}e = -\frac{13}{20}e$.

107. (i) may be proved by differentiation; it is a case of the general theorem that $H_n(x, y, z)/r^{2n+1}$ satisfies Laplace's equation when $H_n(x, y, z)$ does so; where $r = \sqrt{(x^2 + y^2 + z^2)}$, and H_n is a polynomial of degree n. (ii) Assume that the potential inside is Axy, then that outside is $Axya^5/r^5$. Hence, just inside, $\partial V/\partial r = 2V/r$ (since V is $r^2 \times$ a function independent of r); just outside, $\partial V/\partial r = -3V/r$. Then $4\pi\omega = (\partial V/\partial r)$ inside $- (\partial V/\partial r)$ outside $= 5V/r = 5Axy/a$. The conditions are all satisfied if we take $A = 4\pi a/5$.

108. (i) Try $V = f(\lambda)$; $\partial V/\partial x = f'(\lambda)\partial\lambda/\partial x$; $\partial^2 V/\partial x^2 = f'(\lambda)\partial^2\lambda/\partial x^2 + f''(\lambda)(\partial\lambda/\partial x)^2$. $\Delta V = 0$, if $(\partial^2\lambda/\partial x^2 + \partial^2\lambda/\partial y^2 + \partial^2 y/\partial z^2)f'(\lambda) + \{(\partial\lambda/\partial x)^2 + (\partial\lambda/\partial y)^2 + (\partial\lambda/\partial z)^2\}f''(\lambda) = 0$. If the ratio referred to in the question is $F(\lambda)$, then ΔV will be zero if $F(\lambda)f'(\lambda) + f''(\lambda) = 0$, from which $f(\lambda)$ can be found. (ii) Thus, if $\lambda = x^2 + y^2 + z^2$, we have $F(\lambda) = 6/(4x^2 + 4y^2 + 4z^2) = \frac{3}{2}\lambda^{-1}$. Hence $f''(\lambda)/f'(\lambda) = -\frac{3}{2}\lambda^{-1}$, $\log f'(\lambda) = -\frac{3}{2}\log(\lambda/C)$, $f(\lambda) = A\lambda^{-\frac{1}{2}} + B = A/\sqrt{(x^2 + y^2 + z^2)} + B$.

109. $x^2 + y^2 + z^2 = (z + 2\lambda)^2$, $2\lambda = r - z$. Hence, as in Ex. 108, $F(\lambda) = 1/\lambda$, $\log f'(\lambda) = \log(A/\lambda)$, $f(\lambda) = A \log\lambda + B$. The second family is found by taking the negative root, say $2\mu = -r - z$.

110. If the magnetic potential is φ we may take (cf. Ex. 107): $(r > b)$, $\varphi_3 = -Hx + Cx/r^2$; $(b > r > a)$, $\varphi_2 = Ax + Bx/r^2$; $(r < a)$, $\varphi_1 = Dx$; where A, B, C, D are constants, of which D is to be determined. The conditions are: φ is continuous, and the radial component of the induction is continuous. Hence, at $r = b$, $-H + C/b^2 = A + B/b^2$, and $\mu(A/b - B/b^3) = -H/b - C/b^3$; at $r = a$, $A + B/a^2 = D$, and $\mu(A/a - B/a^3) = D/a$. The first two of these four equations give $A + B/b^2 + \mu(A - B/b^2) = -2H$; and the last two give A and B in terms of D, whence D is found.

111. It follows from the theory of p. 137 that, when the cavity is made, the force $\mathbf{H}$ is augmented by the force due to a layer of surface density on the small

sphere, equal to the inward normal component of $\mathbf{I}$. We may suppose $\mathbf{I}$ to be in the direction of Oz, and take O at the centre of the cavity; the density is then $-Iz/r$. The force at the centre due to this is, by symmetry, along Oz, and is $\iint(-Iz/r)(-z/r)(1/r^2)\,dS = 4\pi I/3$. Hence, if $\mathbf{H}_0$, $\mathbf{H}_1$ are the original and final values of $\mathbf{H}$, we have $\mathbf{H}_1 = \mathbf{H}_0 + 4\pi\mathbf{I}/3 = \mathbf{H}_0 + \frac{1}{3}(\mathbf{B} - \mathbf{H}_0)$.

112. Let the face be the plane $z = 0$, and let the magnet be placed at $(0, 0, d)$. Potential due to magnet is $M(z - d)/R_1^3$, where $R_1^2 = x^2 + y^2 + (z-d)^2$. We add to this a potential continuous at $z = 0$, and regular everywhere else; and so chosen that the whole normal induction is continuous at $z = 0$. We may take the added potential to be $A(z + d)/R_2^3$ in the space $z > 0$, where $R_2^2 = x^2 + y^2 + (z + d)^2$; and $-A(z - d)/R_1^3$ in $z < 0$. Then, in $z > 0$, $\partial\varphi/\partial z = \{x^2 + y^2 - 2(z - d)^2\} M/R_1^5 + \{x^2 + y^2 - 2(z + d)^2\} A/R_2^5$; and, in $z < 0$, $\partial\varphi/\partial z = \{x^2 + y^2 - 2(z - d)^2\}(M - A)/R_1^5$. At $z = 0$, we are to have the former of these $= \mu$ times the latter; or, $M + A = \mu(M - A)$, so that $A = M(\mu - 1)/(\mu + 1)$. Then, at $(0, 0, d)$, part of $M\partial^2\varphi/\partial z^2$ due to the A term is $M \,.\, 6A/(z + d)^4 = \frac{3}{8}MA/d^4$.

113. Potential, within and near the sphere, due to pole alone, is $m\{x^2 + y^2 + (z - f)^2\}^{-\frac{1}{2}} = m/f + mz/f^2$ approx. As in Ex. 112, we have to add a potential continuous at the surface of the sphere, and regular elsewhere; say φ_2 (outside) $= Az/r^3$, φ_1 (inside) $= Az/a^3$. Then, at the surface (cf. Ex. 107), $\partial\varphi_0/\partial r = mz/af^2$, $\partial\varphi_1/\partial r = Az/a^4$, $\partial\varphi_2/\partial r = -2Az/a^4$; and $\mu(\partial\varphi_0/\partial r + \partial\varphi_1/\partial r) = \partial\varphi_0/\partial r + \partial\varphi_2/\partial r$. Hence $A = -\{(\mu - 1)/(\mu + 2)\}\,ma^3/f^2$; and force on pole $= m\partial\varphi_2/\partial z = mA(r^2 - 3z^2)/r^5 = -2Am/f^3$.

114. Energy $= \frac{1}{2}QV$ ergs, where Q, V are in e.m.u. or e.s.u. In accumulator, $Q = 100 \times \frac{1}{10} \times 60 \times 60$ e.m.u., $V = 2 \times 10^8$ e.m.u., $QV = 72 \times 10^{11}$ ergs. In condenser, $V = 10^5 \times 10^8$ e.m.u. $= 10^5 \times 10^8 \div (3 \times 10^{10})$ e.s.u., $Q = CV$ in e.s.u., $QV = 10^6V^2 = 10^6 \times (10^6/9) = 10^{12}/9$ ergs. Energy of accumulator/that of condenser $= 64{\cdot}8$.

115. If R is the specific resistance, i the radial current, then, applying Ohm's law to an element of volume $dr\,dS$, we find $(-\partial V/\partial r)\,dr = R(dr/dS)i\,dS$, or $i = -(\partial V/\partial r)/R$. But $i = A/r$, where A is a constant; hence $(-\partial V/\partial r) = AR/r$, $V = AR\log(r_1/r)$. To find A, we have $2 = A \times 33\log_e 60$. Current required $= i \times 2\pi r \times 50 = 4{\cdot}65$ amperes.

116. For an infinite straight current of i e.m.u. we have (p. 126) $H = 2i/r$. Hence, flux through circle $= \int\!\!\int dx\,dy\,2i/(x + h) = 2i\int_0^a \rho\,d\rho \int_0^{2\pi} d\theta/(\rho\cos\theta + h)$ $= 2i\int_0^a \rho\,d\rho \,.\, 2\pi/\sqrt{h^2 - \rho^2}$.

117. As at p. 129, $\mathbf{A} = i\int ds/r$. Take origin at centre of circle, and the point at $(\rho, 0, z)$; then $r^2 = (\rho - a\cos\theta)^2 + a^2\sin^2\theta + z^2 = \rho^2 - 2a\rho\cos\theta + a^2 + z^2$, so that $r = D$. Also $A_x = i\int d(a\cos\theta)/D = -ia\int_0^{2\pi}\sin\theta\,d\theta/D = 0$, since $\int_0^{\pi} = -\int_{\pi}^{2\pi}$; $A_y = i\int d(a\sin\theta)/D = ia\int_0^{2\pi}\cos\theta\,d\theta/D$; $A_z = 0$.

118. Flux (in e.m.u.) at time $t = H \,.\, \pi a^2 \cos\omega t$, so that E.M.F. $= -H \,.\, \pi a^2 \omega \sin\omega t$; current $i = -(H \,.\, \pi a^2\omega/R)\sin\omega t$; average rate of dissipation $= \int Ri^2 dt/T = \frac{1}{2}H^2\pi^2a^4\omega^2/R$.

119. After t sec., let potential across condenser be V volts. Then leaking current $= V \div (2 \times 10^8)$ amperes, and total leak to time $t = \int_0^t V\,dt \div (2 \times 10^8)$ coulombs; residual charge at time $t = t \times 3 \times 10^{-6} - \int_0^t V\,dt \div (2 \times 10^8)$, which must be equal to $V \times 5 \times 10^{-6}$. Differentiate; then $3 \times 10^{-6} - V \div (2 \times 10^8) = 5 \times 10^{-6}\,dV/dt$. Hence $V = (1 - e^{-t/1000})\,600$, which is to be 500; $t = 1000 \log_e 6 = 1792$ sec. $= 30$ min.

120. Flux from a pole of strength m at height h above coil is (downwards) $\int_0^a \{m/(h^2 + r^2)\}\,\{h/\sqrt{h^2 + r^2}\}\,2\pi r\,dr = 2\pi m(1 - h/\sqrt{h^2 + a^2})$. Hence total flux, in question, $= 2\pi m\,[2 - (b + x)/\sqrt{(b + x)^2 + a^2} - (b - x)/\sqrt{(b - x)^2 + a^2}]$. Now $(b + x)/\sqrt{(b + x)^2 + a^2} = (b + x)\,(1/f)\,(1 + 2bx/f^2 + x^2/f^2)^{-\frac{1}{2}}$, the part of which, even in x, is $b/f - 3a^2bx^2/2f^5$. Total flux $= 2\pi M(1/b - 1/f + 3a^2x^2/2f^5)$. E.M.F. $= -dB/dt = -2\pi M(3a^2/2f^5)2x\,dx/dt = -\pi M(3a^2/f^5)nx_0^2 \sin 2nt$; for the current, divide by R.

121. For first part, cf. p. 171. For second part, $L = 4\pi \times 10^6 \times 9\pi \div 50 = 72\pi^2 \times 10^4$ e.m.u. $C = 2 \times 10^{-7}$ farad $= 2 \times 10^{-16}$ e.m.u. $T = 2\pi\sqrt{(LC)} = 236{\cdot}9 \times 10^{-6}$ sec. Frequency $= 4220$ cycles per second.

122. Let PT be the tangent at a point P on the curve. Force on element $i\,ds$ at P, due to N is $i\,ds(m/NP^2)\sin NPT$, perpendicular to the plane; moment of this about $NS = mi\,ds(1/r^2)\,(r\,d\theta/ds)r\sin\theta$; where $r = NP$, $\theta = PNS$. This moment $= mi\,ds\sin\theta\,d\theta/ds = -mi\,d(\cos\theta)$; its integral over the length of the curve $= -mi\,(\cos BNS - \cos ANS)$. Similarly with $(-m)$ at S.

123. $L di/dt + Ri = E\sin 2\pi nt$; $i = A\cos 2\pi nt + B\sin 2\pi nt + Ce^{-Rt/L}$, where C is arbitrary; and A, B are given by $L\,.\,2\pi nB + RA = 0$, with $L(-2\pi nA) + RB = E$. At time $t = x$, $i = 0$; hence, if $C = 0$, $0 = A\cos 2\pi nx + B\sin 2\pi nx$, or $0 = (\cos 2\pi nx + \theta)$, where $\tan\theta = R/(2\pi nL)$; $x = (N\pi + \frac{1}{2}\pi - \theta)/2\pi n$, where N is any integer.

124. If C_1, C_2 are the capacities, then $LC_1d^2e_1/dt^2 + MC_1d^2e_2/dt^2 + e_1 = 0$, and $MC_2\,d^2e_1/dt^2 + NC_2\,d^2e_2/dt^2 + e_2 = 0$. Here $LC_1 = NC_2 = (1/\nu^2)$, say. Try $e_1 = A\cos\omega t$, $e_2 = B\cos\omega t$; on eliminating A, B, we find $(1 - LC_1\omega^2)(1 - NC_2\omega^2) = M^2C_1C_2\omega^4$, or $(1 - \omega^2/\nu^2)^2 = k^2\omega^4/\nu^4$, or $1 - \omega^2/\nu^2 = \pm k\omega^2/\nu^2$, or $\omega^2/\nu^2 = 1/(1 \pm k)$.

125. $LC\,d^2e_1/dt^2 + MC\,d^2e_2/dt^2 + RC\,de_1/dt + e_1 = 0$, and $MC\,d^2e_1/dt^2 + LC\,d^2e_2/dt^2 + RC\,de_2/dt + e_2 = 0$. By addition, $(L + M)C\,d^2(e_1 + e_2)/dt^2 + RC\,d(e_1 + e_2)/dt + (e_1 + e_2) = 0$. Put $e_1 + e_2 = e^{jnt}$; then $(L + M)Cn^2 - RCjn - 1 = 0$; $n = (RCj \pm \sqrt{\alpha^2 + 4MC})/\{2(L + M)C\}$; where $\alpha = \sqrt{4LC - R^2C^2}$. Since M is small, this gives (if $n = n_1 + jn_2$) $n_1 = \alpha(1 + 2MC/\alpha^2 - M/L)/(2LC)$; also, by putting $M = 0$ in this, $2\pi/T = \alpha/(2LC)$. Hence $2\pi/T' = (2\pi/T)$

$\{1 + (M/L)(-1 + T^2/8\pi^2 LC)\}$. The second period is obtained by subtracting instead of adding the original equations; $e_1 + e_2$ has the one period, $e_1 - e_2$ the other; so that e_1, e_2 have the two periods.

126. Let pressure across the circuits be e_1, e_2; each involving the factor $e^{j\omega t}$. If given pressure is e, then $e = e_1 + e_2$. The current i is the same in both circuits. Hence $i\{R_1 + j(\omega L_1 - 1/\omega C_1)\} = e_1$, and $i\{R_2 + j(\omega L_2 - 1/\omega C_2)\} = e_2$, so that, by addition, $i\{(R_1 + R_2) + j(X_1 + X_2)\} = e$; where X_1, X_2 are the two reactances $X_1 = \omega L_1 - 1/(\omega C_1)$, $X_2 = \omega L_2 - 1/(\omega C_2)$. This gives i as at p. 173 (cf. Ex. 127).

127. Here $i_1(R_1 + jX_1) = e$, $i_2(R_2 + jX_2) = e$ (cf. Ex. 126). In this case $i_1 + i_2 = i$. Hence $i = e/(R_1 + jX_1) + e/(R_2 + jX_2)$, or $i = \dfrac{e(R_1 - jX_1)}{R_1^2 + X_1^2} + \dfrac{e(R_2 - jX_2)}{R_2^2 + X_2^2}$. If $e = Ee^{j\omega t}$, where E is real, the real part of this expression for i gives the current for the pressure $E\cos\omega t$, and the coefficient of j in the expression gives the current for the pressure $E\sin\omega t$.

128. If I, E denote root-mean-square values, then before the condenser is inserted, $E_1 = RI$. Afterwards (Ex. 127), $i = e/R + e \div (-j/\omega C) = e(1/R + j\omega C)$; from which, on taking the moduli, we get $I = E_2\sqrt{(1/R^2 + \omega^2 C^2)}$. Hence reduction in the voltage drop $= E_1 - E_2 = IR[1 - 1/\sqrt{(1 + \omega^2 C^2 R^2)}]$; and $(E_1 - E_2)/E_1 = 1 - 1/\sqrt{(1 + \omega^2 C^2 R^2)}$. Thus $\sqrt{(1 + \omega^2 C^2 R^2)} = 20/19$. But $C = 3 \times 10^5$ e.s.u. $= 3 \times 10^5 \div (9 \times 10^{20})$ e.m.u. $= 1/(3 \times 10^{15})$ e.m.u.; $CR = 1/(3 \times 10^{15}) \times (3 \times 10^4) \times 10^9 = 1/100$ sec. Then $\omega CR = \sqrt{39}/19$, $\omega = 33$ cycles per second.

129. See Examples 126, 127. We have here only L_1, C_2. In case (i), $ij(\omega L_1 - 1/\omega C_2) = e$; in case (ii), $i = e/jX_1 + e/jX_2 = (e/j)(X_1 + X_2)/X_1X_2 = (e/j)(\omega L_1 - 1/\omega C_2) \div (-L_1/C_2)$.

130. Let Q be charge of first condenser at time t, $Q_0 - Q$ charge of second; potential of first $= Q/C_1$, of second $(Q_0 - Q)/C_2$. Pressure from first to second $= Q/C_1 - (Q_0 - Q)/C_2$; so that $L\,di/dt + Ri = Q(1/C_1 + 1/C_2) - Q_0/C_2$; and $L\,d^2i/dt^2 + R\,di/dt + i(1/C_1 + 1/C_2) = 0$. If we try $i = e^{-pt}$, we find that the two values of p are equal, viz. $p = R/2L$. Solution is $Ae^{-pt} + Bte^{-pt}$; but $A = 0$, since $i = 0$ when $t = 0$. The differential equation now gives, when t is put equal to 0, $BL = Q_0/C_1$. Current is a maximum when $e^{-pt} - pte^{-pt} = 0$, or $t = 1/p = 2L/R$.

131. Integrate the equation $L\,di/dt + Ri = E$ from $t = 0$ to $t = \tau$; then $Li_\tau + R\int_0^\tau i\,dt = \int_0^\tau E\,dt$. The value of the latter integral when τ becomes indefinitely small, is the *impulse*, P. Now, when t is small, i is of order t, but $\int_0^t i\,dt$ of order t^2, as we easily see from the formula for the case of constant E, viz. $i = A(1 - e^{-Rt/L})$. Hence, passing to the limit, we find $Li = P$, where i is now the initial current due to the impulse. By integrating from 0 to ∞, we find: total charge $= E/R$. (ii) Equations are: $L\,di_1/dt + M\,di_2/dt + R_1 i_1 = E$.

$N\,di_2/dt + M\,di_1/dt + R_2 i_2 = 0$; so that, by preceding argument, $Li_1 + Mi_2 = P$, and $Ni_2 + Mi_1 = 0$; and therefore $(L - M^2/N)i_1 = P$.

132. By Ex. 121, self-inductance L of first coil $= 4\pi \times 10^6 \times 4\pi/100 = 16\pi^2 \times 10^4$ e.m.u. The mutual inductance $M = 4\pi n_1 n_2 A/l$, in e.m.u. $= 64\pi^2 \times 1000$ e.m.u. The self-inductance N of the second coil may be taken as equal to M. Equations are:

$$L\,di_1/dt + M\,di_2/dt + R_1 i_1 = E,\quad M\,di_1/dt + M\,di_2/dt + R_2 i_2 = 0.$$

(1) Integrate latter equation from $t = 0$ to $t = \infty$; then $Mi_1 + R_2\int i_2\,dt = 0$; that is, charge passing $= -Mi_1/R_2 = 1{\cdot}4 \times 10^{-4}$ coulomb. (2) In same equation, if i_1 varies as $e^{j\omega t}$, $M\,di_2/dt + R_2 i_2 = -Mj\omega i_1$, or $i_2 = -\{Mj\omega/(R_2 + Mj\omega)\}\,i_1$, or, for root-mean-square values, $I_2 = I_1 M\omega/\sqrt{(R_2{}^2 + M^2\omega^2)} = {\cdot}066$ amperes.

133. Let potential of plate joined to B be 0, and charge of other plate Q at time t. Then total current from A to $B = -dQ/dt = i$, and pressure $= Q/C$. If partial currents are i_1, i_2, then $L_1\,di_1/dt + R_1 i_1 = Q/C$, $L_2\,di_2/dt + R_2 i_2 = Q/C$, $i_1 + i_2 = i$. When we integrate over the whole time of discharge, the terms in L_1, L_2 disappear, and we get $R_1\int i_1\,dt = R_2\int i_2\,dt$. Hence $\int i_1\,dt : \int i_2\,dt$: initial charge $= R_2 : R_1 : R_1 + R_2$.

134. $L\,di_1/dt + L\,di_2/dt + Ri_1 = E\sin nt$, $L\,di_1/dt + L\,di_2/dt + Ri_2 = 0$. Add and subtract; then $2L\,d(i_1 + i_2)/dt + R(i_1 + i_2) = E\sin nt$, and $R(i_1 - i_2) = E\sin nt$. These give $i_1 + i_2$ and $i_1 - i_2$. We find $2i_1/E = \{R/(R^2 + 4L^2n^2) + 1/R\}\sin nt - \{2Ln/(R^2 + 4L^2n^2)\}\cos nt$. Mean rate of energy supply $= \frac{1}{2}$ coefficient of $\sin^2 nt$ in $i_1 E\sin nt$.

135. (i) As in § 11, p. 223, V satisfies the equation $\partial^2 V/\partial t^2 = c^2\Delta V$. Hence E_x, E_y, E_z, in the forms given, all satisfy a similar equation; also $\partial E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z = 0$; and there are just the four equations obtained by eliminating H_x, H_y, H_z from Maxwell's equations.

(ii) The surface condition for a perfect conductor is that the electric force must be in the direction of the normal, or that $\dfrac{\partial^2 V}{\partial x\,\partial z} : \dfrac{\partial^2 V}{\partial y\,\partial z} : (-)\left(\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2}\right) = x : y : z$, at $r = a$. Since V is a function of r, we find $\dfrac{\partial^2 V}{\partial x\,\partial z} = \dfrac{xz}{r}\dfrac{d}{dr}\left(\dfrac{1}{r}\dfrac{dV}{dr}\right)$, $\dfrac{\partial^2 V}{\partial y\,dz} = \dfrac{yz}{r}\dfrac{d}{dr}\left(\dfrac{1}{r}\dfrac{dV}{dr}\right)$, $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = -\dfrac{z^2}{r}\dfrac{d}{dr}\left(\dfrac{1}{r}\dfrac{dV}{dr}\right) - \dfrac{1}{r}\dfrac{dV}{dr} - \dfrac{\nu^2}{c^2}V$; using the fact that $\dfrac{d^2 V}{dr^2} + \dfrac{2}{r}\dfrac{dV}{dr} + \dfrac{\nu^2}{c^2}V = 0$. Hence the condition is that $\dfrac{1}{r}\dfrac{dV}{dr} + \dfrac{\nu^2}{c^2}V = 0$, at $r = a$.

136. The electric vector is parallel to the axis, and the lines of magnetic force are circles round the axis. As at p. 126, $2\pi rH = \int 4\pi i\,.\,2\pi r\,dr$, or $\partial/\partial r(rH) = 4\pi ri$. Again, apply Faraday's law (p. 141) to the rectangle with two sides parallel to the axis, at distances r and $r + dr$ from it, and with its other sides at unit distance apart. This gives $\{E + (\partial E/\partial r)dr\} - E = \mu(\partial H/\partial t)dr$.

137. The two equations of Ex. 136 become (since $i = \sigma E$) $4\pi ri = \partial/\partial r(Hr)$, $\partial i/\partial r = j\mu\sigma\omega H$; and, on elimination of H, $\partial/\partial r(r\partial i/\partial r) = (4\pi j\mu\sigma\omega)ri$. Write ε^2 for the small fraction $\pi\mu\sigma\omega a^2$, and put u for $\varepsilon r/a$. Then $\partial/\partial u(u\,\partial i/\partial u) = 4jui$. The solution of this equation can be written as a series of ascending powers of u^2 of which, since u is small, we need only retain the first two terms. Thus $i = (1 + ju^2)Ae^{j\omega t} = Ae^{j\omega t}(1 + \pi j\mu\sigma\omega r^2) = Ae^{j\omega t}(1 + j\varepsilon^2r^2/a^2)$. If the root-mean-square current at the distance r is I, this gives $I^2 = \frac{1}{2}A^2(1 + \varepsilon^4r^4/a^4)$; and the mean rate of development of heat is I^2/σ per c.c. per second. Hence rate of development of heat per unit length of the whole wire $= \int_0^a (I^2/\sigma)2\pi r\,dr$ $= (A^2\pi a^2/2\sigma)(1 + \frac{1}{3}\varepsilon^4)$. Again, integral current $= \int_0^a 2\pi ri\,dr = Ae^{j\omega t}\,.\,\pi a^2$ $(1 + \frac{1}{2}j\varepsilon^2)$; and, if r.m.s. integral current $= I_0$, then $I_0^2 = \frac{1}{2}A^2\pi^2a^4(1 + \frac{1}{4}\varepsilon^4)$. Hence heat per unit length and time $= (I_0^2/\pi\sigma a^2)(1 + \frac{1}{12}\varepsilon^4)$.

INDEX